Zadatak 49. Zadana je jednadžba hiperbole $x^2 - y^2 = a^2$. Točkom T hiperbole povučeni su pravci paralelni asimptotama. Dokaži da površina četverokuta omeđenog tim pravcima i asimptotama ne ovisi o izboru točke T.

Rješenje.

$$
\begin{aligned}
& A(x_0, y_0) \\
a_1 & \ \dots\ y = x, \quad k = 1 \\
a_2 & \ \dots\ y = -x, \quad k = -1 \\
p_1 \| a_1 & \ \dots\ y = x + l_1 \quad (k = 1) \\
p_2 \| a_2 & \ \dots\ y = -x + l_1 \quad (k = -1) \\
\{A\} \in p_1 & \Longrightarrow y_0 = x_0 + l_1, \ \ l_1 = y_0 - x_0 \\
\{A\} \in p_2 & \Longrightarrow y_0 = -x_0 + l_2, \ \ l_2 = y_0 + x_0 \\
a_1 & \ \dots\ y = x, \quad k = 1 \\
a_2 & \ \dots\ y = -x, \quad k = -1 \\
p_1 & \ \dots\ y = x + y_0 - x_0 \\
p_2 & \ \dots\ y = -x + y_0 + x_0 \\
& \boxed{A(x_0, y_0)} \\
B = a_1 \cap p_2 & \Longrightarrow x = -x + y_0 + x_0 \\
& \quad x = \frac{y_0 + x_0}{2} \Longrightarrow \boxed{B\left(\frac{y_0 + x_0}{2}, \frac{y_0 + x_0}{2}\right)} \\
C = a_2 \cap p_1 & \Longrightarrow -x = x + y_0 - x_0 \\
& \quad x = \frac{x_0 - y_0}{2} \Longrightarrow \boxed{C\left(\frac{x_0 - y_0}{2}, -\frac{x_0 - y_0}{2}\right)} \\
\boxed{P} & = d(A, B) \cdot d(A, C) \\
& = \sqrt{\left(\frac{y_0 + x_0}{2} - x_0\right)^2 + \left(\frac{y_0 + x_0}{2} - y_0\right)^2} \cdot \sqrt{\left(\frac{x_0 - y_0}{2} - x_0\right)^2 + \left(\frac{x_0 - y_0}{2} - y_0\right)^2} \\
& = \sqrt{\left(\frac{y_0 - x_0}{2}\right)^2 + \left(\frac{x_0 - y_0}{2}\right)^2} \cdot \sqrt{\left(\frac{x_0 + y_0}{2}\right)^2 + \left(\frac{x_0 + y_0}{2}\right)^2} \\
& = \sqrt{\left(\frac{y_0 - x_0}{2}\right)^2 + \left(\frac{y_0 - x_0}{2}\right)^2} \cdot \sqrt{\left(\frac{y_0 + x_0}{2}\right)^2 + \left(\frac{y_0 + x_0}{2}\right)^2} \\
& = \sqrt{2\left(\frac{y_0 - x_0}{2}\right)^2} \cdot \sqrt{2\left(\frac{y_0 + x_0}{2}\right)^2} \\
& = \frac{1}{2}(y_0 - x_0) \cdot (y_0 + x_0) \\
& = \frac{1}{2}(y_0^2 - x_0^2) \quad \left(T(x_0, y_0) \in H \Longrightarrow y_0^2 - x_0^2 = a^2\right) \\
& \boxed{= \frac{1}{2}a^2}
\end{aligned}
$$