

Zadatak 46.

Dokaži da se kružnice $x^2 + y^2 - 2mx - 2ny - m^2 + n^2 = 0$ i $x^2 + y^2 - 2nx + 2my + m^2 - n^2 = 0$ sijeku pod pravim kutom.

Rješenje.

$$\begin{aligned} -2m &= -2p_1 \implies p_1 = m, \quad -2n = -2q_1 \implies q_1 = n, \\ n^2 - m^2 &= m^2 + n^2 - r_1^2 \implies r_1^2 = m^2 - 2n = -2p_2 \implies p_2 = n, \\ 2m &= -2q_2 \implies q_2 = -m, \\ m^2 - n^2 &= n^2 + m^2 - r_2^2 \implies r_2^2 = n^2 \end{aligned}$$

$$\begin{aligned} x^2 + y^2 - 2mx - 2ny - m^2 + n^2 &= 0 \\ x^2 + y^2 - 2nx + 2my + m^2 - n^2 &= 0 \end{aligned}$$

$$\begin{aligned} 2x(n-m) - 2y(n+m) - 2m^2 + 2n^2 &= 0 \\ x(n-m) - y(n+m) + (n-m)(n+m) &= 0 \\ y &= \frac{n-m}{n+m}x + n - m \end{aligned}$$

$$\begin{aligned} x^2 + \left(\frac{n-m}{n+m}\right)x^2 + 2\frac{(n-m)^2}{n+m}x + (n-m)^2 - 2mx \\ - \frac{2n(n-m)}{n+m}x - 2n^2 + 2nm - m^2 + n^2 &= 0 \\ x^2 \frac{n^2 + 2nm + m^2 + n^2 - 2nm + m^2}{(n+m)^2} \\ + x \frac{2n^2 - 4nm + 2m^2 - 2mn - 2m^2 - 2n^2 + 2nm}{n+m} \\ + n^2 - 2nm + m^2 - n^2 + 2nm - m^2 &= 0 \\ x^2 \frac{2(n^2 + m^2)}{(n+m)^2} + x \frac{-4nm}{n+m} &= 0 \\ x^2(n^2 + m^2) - 2nm(n+m)x &= 0 \\ x(x(n^2 + m^2) - 2nm(n+m)) &= 0 \\ x_1 = 0, \quad y_1 = n - m & \end{aligned}$$

Dovoljna je jedna točka, $T(0, n - m)$.

$$\begin{aligned} (0-m)(x-m) + (n-m-n)(y-n) &= m^2 \\ -mx + m^2 - my + mn &= m^2 \\ -x - y + n &= 0 \\ y &= -x + n \\ (0-n)(x-n) + (n-m+m)(y+m) &= n^2 \\ -nx + n^2 + ny + nm &= n^2 \\ -x + y + m &= 0 \\ y &= x - m \end{aligned}$$

Kružnice se sijeku pod pravim kutom jer su tangente okomite, $k_1 = -\frac{1}{k_2}$.