Zadatak 8. Dokaži da u svakom trokutu vrijede relacije

1) $\dfrac{\sin(\alpha-\beta)}{\sin(\alpha+\beta})=\dfrac{a^2-b^2}{c^2}$;

2) $\dfrac{\operatorname{tg}\alpha}{\operatorname{tg}\beta}=\dfrac{c^2+a^2-b^2}{b^2+c^2-a^2}$;

3) $\dfrac{\sin\dfrac{\alpha}{2}}{\cos\dfrac{\beta}{2}\cos\dfrac{\gamma}{2}}=\dfrac{a}{s}$;

4) $\dfrac{\cos\dfrac{\beta-\gamma}{2}}{\sin\dfrac{\alpha}{2}}=\dfrac{b+c}{a}$.

Rješenje.

1) $\dfrac{\sin(\alpha-\beta)}{\sin(\alpha+\beta})=\dfrac{a^2-b^2}{c^2}$. Po Mollweidovim formulama

$$\left.\begin{aligned}\frac{a+b}{c}&=\frac{\cos\frac{\alpha-\beta}{2}}{\sin\frac{\gamma}{2}}\\ \frac{a-b}{c}&=\frac{\sin\frac{\alpha-\beta}{2}}{\cos\frac{\gamma}{2}}\end{aligned}\right\}$$

$$\frac{2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha-\beta}{2}}{\sin(180^\circ-\gamma)}=\frac{(a-b)(a+b)}{c^2}$$

$$\frac{2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha-\beta}{2}}{\sin 180^\circ\cos\gamma-\cos 180^\circ\sin\gamma)}=\frac{(a-b)(a+b)}{c^2}$$

$$\frac{2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha-\beta}{2}}{2\sin\frac{\gamma}{2}\cos\frac{\gamma}{2}}=\frac{(a-b)(a+b)}{c^2}$$

$$\frac{a-b}{c}\cdot\frac{a+b}{c}=\frac{a-b}{c}\cdot\frac{a+b}{c}$$

2) $\dfrac{\operatorname{tg}\alpha}{\operatorname{tg}\beta}=\dfrac{c^2+a^2-b^2}{b^2+c^2-a^2}$. Po kosinusovom poučku vrijedi

$$\cos\beta=\frac{c^2+a^2-b^2}{2ac},\qquad \cos\alpha=\frac{b^2+c^2-a^2}{2bc}$$

odakle slijedi

$$c^2+a^2-b^2=2ac\cos\beta,\qquad b^2+c^2-a^2=2bc\cos\alpha$$

$$\frac{\operatorname{tg}\alpha}{\operatorname{tg}\beta}=\frac{2ac\cos\beta}{2bc\cos\alpha}$$

$$\frac{\frac{\sin\alpha}{\cos\alpha}}{\frac{\sin\beta}{\cos\beta}}=\frac{2ac\cos\beta}{2bc\cos\alpha}$$

$$\frac{\sin\alpha\cos\beta}{\cos\alpha\sin\beta}=\frac{a\cos\beta}{b\cos\alpha}$$

$$\frac{\sin\alpha}{\sin\beta}=\frac{a}{b}\quad\text{(sinusov poučak)}$$

3) $$\frac{\sin\frac{\alpha}{2}}{\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}=\frac{a}{s};$$

$$s=a+b+c$$

$$s=4r\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\gamma 2$$

$$r=\frac{a}{2\sin\alpha}\implies a=2r\sin\alpha$$

$$\frac{a}{s}=\frac{2r\sin\alpha}{4r\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}=\frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{2\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}=\frac{\sin\frac{\alpha}{2}}{\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}$$

4) $$\frac{\sin\frac{\alpha}{2}}{\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}=\frac{a}{s}\quad\text{(Mollweidova formula)}$$

$$\frac{2\cos\frac{\beta-\gamma}{2}\sin\frac{\beta+\gamma}{2}}{2\sin\frac{\alpha}{2}\sin\frac{\beta+\gamma}{2}}=\left(\frac{\beta+\gamma}{2}=\frac{180^\circ-\alpha}{2}\right)=\frac{\sin\gamma+\sin\beta}{2\sin\frac{\alpha}{2}\sin\left(90^\circ-\frac{\alpha}{2}\right)}$$

$$=\frac{\sin\gamma+\sin\beta}{2\sin\frac{\alpha}{2}\left(\sin 90^\circ\cos\frac{\alpha}{2}-\cos 90^\circ\sin\frac{\alpha}{2}\right)}$$

$$=\frac{\sin\gamma+\sin\beta}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}=\frac{\sin\gamma+\sin\beta}{\sin\alpha}$$

$$=\left(2r=\frac{a}{\sin\alpha}=\frac{b}{\sin\beta}=\frac{c}{\sin\gamma}\right)=\frac{\frac{c}{2r}+\frac{b}{2r}}{\frac{a}{2r}}=\frac{c+b}{a}$$