Zadatak 9. Dokaži da u svakom trokutu vrijede relacije

1) $\dfrac{a\sin\alpha+b\sin\beta+c\sin\gamma}{4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}=\dfrac{a^2+b^2+c^2}{a+b+c}$;

2) $a\sin(\beta-\gamma)+b\sin(\gamma-\alpha)+c\sin(\alpha-\beta)=0$;

3) $\dfrac{a^2\sin(\beta-\gamma)}{\sin\alpha}+\dfrac{b^2\sin(\gamma-\alpha)}{\sin\beta}+\dfrac{c^2\sin(\alpha-\beta)}{\sin\gamma}=0$;

4) $\dfrac{1}{a}\cos^2\dfrac{\alpha}{2}+\dfrac{1}{b}\cos^2\dfrac{\beta}{2}+\dfrac{1}{c}\cos^2\dfrac{\gamma}{2}=\dfrac{s^2}{abc}$.

Rješenje. 1) $\dfrac{a\sin\alpha+b\sin\beta+c\sin\gamma}{4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}=\dfrac{a^2+b^2+c^2}{a+b+c}$

$$\left(\sin\alpha+\sin\beta+\sin\gamma=4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2},\quad 2r=\frac{a}{\sin\alpha}=\frac{b}{\sin\beta}=\frac{c}{\sin\gamma}\right)$$

$$\frac{a\sin\alpha+b\sin\beta+c\sin\gamma}{4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}=\frac{a\cdot\frac{a}{2r}+b\cdot\frac{b}{2r}+c\cdot\frac{c}{2r}}{\sin\alpha+\sin\beta+\sin\gamma}=\frac{\frac{a^2+b^2+c^2}{2r}}{\frac{a}{2r}+\frac{b}{2r}+\frac{c}{2r}}=\frac{a^2+b^2+c^2}{a+b+c}$$

2) $a\sin(\beta-\gamma)+b\sin(\gamma-\alpha)+c\sin(\alpha-\beta)=0$

Iz Mollweidove formule imamo:

$$\frac{b-c}{a}=\frac{\sin\frac{\beta-\gamma}{2}}{\cos\frac{\alpha}{2}}\implies$$

$$a\sin\frac{\beta-\gamma}{2}=(b-c)\cos\frac{\alpha}{2}\quad\Big/\cdot 2\cos\frac{\beta-\gamma}{2}$$

$$2a\sin\frac{\beta-\gamma}{2}\cdot\cos\frac{\beta-\gamma}{2}=2(b-c)\cos\frac{\alpha}{2}\cdot\cos\frac{\beta-\gamma}{2}$$

$$a\sin(\beta-\gamma)=2(b-c)\cos\frac{\alpha}{2}\cdot\cos\frac{\beta-\gamma}{2}$$

$$a\sin(\beta-\gamma)=2(b-c)\cdot\frac{1}{2}\left[\cos\left(\frac{\alpha}{2}+\frac{\beta-\gamma}{2}\right)+\cos\left(\frac{\alpha}{2}-\frac{\beta-\gamma}{2}\right)\right]$$

$$a\sin(\beta-\gamma)=(b-c)\cdot\left(\cos\frac{\alpha+\beta-\gamma}{2}+\cos\frac{\alpha-\beta+\gamma}{2}\right)$$

$$a\sin(\beta-\gamma)=(b-c)\cdot\left(\cos\frac{180^\circ-\gamma-\gamma}{2}+\cos\frac{180^\circ-\beta-\beta}{2}\right)$$

$$a\sin(\beta-\gamma)=(b-c)\cdot[\cos(90^\circ-\gamma)+\cos(90^\circ-\beta)]$$

$$a\sin(\beta-\gamma)=(b-c)\cdot(\sin\gamma+\sin\beta)$$

$$a\sin(\beta-\gamma)=(b-c)\cdot\left(\frac{c}{2r}+\frac{b}{2r}\right)$$

$$a\sin(\beta-\gamma)=\frac{(b-c)\cdot(b+c)}{2r}$$

$$a\sin(\beta-\gamma)=\frac{b^2-c^2}{2r}$$

Analogno se dobije $b\sin(\gamma-\alpha)=\dfrac{c^2-a^2}{2r}$, $c\sin(\alpha-\beta)=\dfrac{a^2-b^2}{2r}$, pa imamo

$$\begin{aligned} a\sin(\beta-\gamma)+b\sin(\gamma-\alpha)+c\sin(\alpha-\beta)&=\frac{b^2-c^2}{2r}+\frac{c^2-a^2}{2r}+\frac{a^2-b^2}{2r}\\ &=\frac{b^2-c^2+c^2-a^2+a^2-b^2}{2r}=0 \end{aligned}$$

3) $\dfrac{a^2\sin(\beta-\gamma)}{\sin\alpha}+\dfrac{b^2\sin(\gamma-\alpha)}{\sin\beta}+\dfrac{c^2\sin(\alpha-\beta)}{\sin\gamma}=0$

Iz zadatka pod **2)** dobije se $a^2\sin(\beta-\gamma)=a\dfrac{b^2-c^2}{2r}$, $b^2\sin(\gamma-\alpha)=b\dfrac{c^2-a^2}{2r}$, $c^2\sin(\alpha-\beta)=c\dfrac{a^2-b^2}{2r}$, pa imamo:

$$\frac{a^2\sin(\beta-\gamma)}{\sin\alpha}+\frac{b^2\sin(\gamma-\alpha)}{\sin\beta}+\frac{c^2\sin(\alpha-\beta)}{\sin\gamma}=\frac{a\frac{b^2-c^2}{2r}}{\frac{a}{2r}}+\frac{b\frac{c^2-a^2}{2r}}{\frac{b}{2r}}+\frac{c\frac{a^2-b^2}{2r}}{\frac{c}{2r}}=0$$

4)

$$\begin{aligned} \frac{1}{a}\cos^2\frac{\alpha}{2}+\frac{1}{b}\cos^2\frac{\beta}{2}+\frac{1}{c}\cos^2\frac{\gamma}{2}&=\frac{1}{a}\frac{1+\cos\alpha}{2}+\frac{1}{b}\frac{1+\cos\beta}{2}+\frac{1}{c}\frac{1+\cos\gamma}{2}\\ &=\frac{bc(1+\cos\alpha)+ac(1+\cos\beta)+ab(1+\cos\gamma)}{2abc}\\ &=\frac{bc+ac+ab+bc\cos\alpha+ac\cos\beta+ab\cos\gamma}{2abc}\\ &=\frac{bc+ac+ab+bc\cdot\frac{b^2+c^2-a^2}{2bc}+ac\cdot\frac{a^2+c^2-b^2}{2ac}+ab\cdot\frac{a^2+b^2-c^2}{2ab}}{2abc}\\ &=\frac{\frac{2bc+2ac+2ab+b^2+c^2-a^2+a^2+c^2-b^2+a^2+b^2-c^2}{2}}{2abc}\\ &=\frac{(a+b+c)^2}{4abc}=\frac{s^2}{abc} \end{aligned}$$