Zadatak 3. Dokaži sljedeće identitete:

1) $\sin^2 x \cdot \sin^2 y + \sin^2 x \cdot \cos^2 y + \cos^2 x = 1$;

2) $\cos^2 x + \sin^2 x \cdot \cos^2 x + \sin^4 x = 1$;

3) $3\sin^4 x - 2\sin^6 x = 1 - 3\cos^4 x + 2\cos^6 x$;

4) $(\mathrm{tg}^2 x - \sin^2 x) \cdot \mathrm{ctg}^2 x = \sin^2 x$;

5) $(1 + \mathrm{ctg}^2 x)(1 - \sin^2 x) = \mathrm{ctg}^2 x$;

6) $(\sin x + \mathrm{tg}\, x)(\cos x + \mathrm{ctg}\, x)$
$= (1 + \sin x)(1 + \cos x)$;

7) $1 - \dfrac{1}{1 + \mathrm{tg}^2 x} = \dfrac{1}{1 + \mathrm{ctg}^2 x}$;

8) $\dfrac{\mathrm{tg}\, x}{1 - \mathrm{tg}^2 x} \cdot \dfrac{\mathrm{ctg}^2 x - 1}{\mathrm{ctg}\, x} = 1$;

9) $\dfrac{\sin^2 x}{\sin x - \cos x} - \dfrac{\sin x + \cos x}{\mathrm{tg}^2 x - 1} = \sin x + \cos x$;

10) $\dfrac{\sin^3 x - \cos^3 x}{1 + \sin x \cos x} = \sin x - \cos x$.

Rješenje. **1)**

$$\begin{aligned}
\sin^2 x \cdot \sin^2 y + \sin^2 x \cdot \cos^2 y + \cos^2 x &= 1\\
\sin^2 x \cdot \underbrace{(\sin^2 y + \cos^2 y)}_{1} + \cos^2 x &= 1\\
\sin^2 x + \cos^2 x &= 1\\
1 &= 1
\end{aligned}$$

2)

$$\begin{aligned}
\cos^2 x + \sin^2 x \cdot \cos^2 x + \sin^4 x &= 1\\
\cos^2 x + \sin^2 x \cdot (\cos^2 x + \sin^2 x) &= 1\\
\cos^2 x + \sin^2 x &= 1\\
1 &= 1
\end{aligned}$$

3)

$$\begin{aligned}
3\sin^4 x - 2\sin^6 x &= 1 - 3\cos^4 x + 2\cos^6 x\\
3\sin^4 x - 2\sin^6 x &= (1 - \cos^2 x)(1 + \cos^2 x) - 2\cos^4 x(1 - \cos^2 x)\\
3\sin^4 x - 2\sin^6 x &= \sin^2 x \cdot (1 + \cos^2 x) - 2\cos^4 x \cdot \sin^2 x\\
\sin^4 x \cdot (3 - 2\sin^2 x) &= \sin^2 x \cdot (1 + \cos^2 x - 2\cos^4 x)\\
\sin^4 x \cdot [1 + 2(1 - \sin^2 x)] &= \sin^2 x \cdot (1 - \cos^4 x + \cos^2 x - \cos^4 x)\\
\sin^4 x \cdot (1 + 2\cos^2 x) &= \sin^2 x \cdot [(1 - \cos^2 x)(1 + \cos^2 x) + \cos^2 x(1 - \cos^2 x)]\\
\sin^4 x \cdot (1 + 2\cos^2 x) &= \sin^2 x \cdot [\sin^2 x \cdot (1 + \cos^2 x) + \cos^2 x \cdot \sin^2 x]\\
\sin^4 x \cdot (1 + 2\cos^2 x) &= \sin^4 x \cdot (1 + \cos^2 x + \cos^2 x)\\
\sin^4 x \cdot (1 + 2\cos^2 x) &= \sin^4 x \cdot (1 + 2\cos^2 x)
\end{aligned}$$

4)

$$(\mathrm{tg}^2 x - \sin^2 x) \cdot \mathrm{ctg}^2 x = \sin^2 x$$
$$\left(\frac{\sin^2 x}{\cos^2 x} - \sin^2 x\right) \cdot \frac{\cos^2 x}{\sin^2 x} = \sin^2 x$$
$$\frac{\sin^2 x - \sin^2 x \cdot \cos^2 x}{\cos^2 x} \cdot \frac{\cos^2 x}{\sin^2 x} = \sin^2 x$$
$$\frac{\sin^2 x(1 - \cos^2 x)}{\sin^2 x} = \sin^2 x$$
$$1 - \cos^2 x = \sin^2 x$$
$$\sin^2 x = \sin^2 x$$

5)

$$(1 + \mathrm{ctg}^2 x)(1 - \sin^2 x) = \mathrm{ctg}^2 x$$
$$\left(1 + \frac{\cos^2 x}{\sin^2 x}\right) \cdot \cos^2 x = \mathrm{ctg}^2 x$$
$$\frac{\sin^2 x + \cos^2 x}{\sin^2 x} \cdot \cos^2 x = \mathrm{ctg}^2 x$$
$$\frac{1}{\sin^2 x} \cdot \cos^2 x = \mathrm{ctg}^2 x$$
$$\mathrm{ctg}^2 x = \mathrm{ctg}^2 x$$

6)

$$(\sin x + \mathrm{tg}\, x)(\cos x + \mathrm{ctg}\, x) = (1 + \sin x)(1 + \cos x)$$
$$\left(\sin x + \frac{\sin x}{\cos x}\right)\left(\cos x + \frac{\cos x}{\sin x}\right) = (1 + \sin x)(1 + \cos x)$$
$$\frac{\sin x \cdot \cos x + \sin x}{\cos x} \cdot \frac{\cos x \cdot \sin x + \cos x}{\sin x} = (1 + \sin x)(1 + \cos x)$$
$$\frac{\sin x \cdot (1 + \cos x)}{\cos x} \cdot \frac{\cos x \cdot (1 + \sin x)}{\sin x} = (1 + \sin x)(1 + \cos x)$$
$$(1 + \cos x)(1 + \sin x) = (1 + \sin x)(1 + \cos x)$$

7)

$$1 - \frac{1}{1 + \mathrm{tg}^2 x} = \frac{1}{1 + \mathrm{ctg}^2 x}$$
$$\frac{1 + \mathrm{tg}^2 x - 1}{1 + \mathrm{tg}^2 x} = \frac{1}{1 + \mathrm{ctg}^2 x}$$
$$\frac{\mathrm{tg}^2 x}{1 + \mathrm{tg}^2 x} = \frac{1}{1 + \mathrm{ctg}^2 x}$$
$$\frac{\frac{1}{\mathrm{ctg}^2 x}}{1 + \frac{1}{\mathrm{ctg}^2 x}} = \frac{1}{1 + \mathrm{ctg}^2 x}$$
$$\frac{\frac{1}{\mathrm{ctg}^2 x}}{\frac{\mathrm{ctg}^2 x + 1}{\mathrm{ctg}^2 x}} = \frac{1}{1 + \mathrm{ctg}^2 x}$$
$$\frac{1}{\mathrm{ctg}^2 x + 1} = \frac{1}{1 + \mathrm{ctg}^2 x}$$

8)

$$\frac{\operatorname{tg} x}{1-\operatorname{tg}^2 x} \cdot \frac{\operatorname{ctg}^2 x - 1}{\operatorname{ctg} x} = 1$$

$$\frac{\operatorname{tg} x}{1-\operatorname{tg}^2 x} \cdot \frac{\frac{1}{\operatorname{tg}^2 x} - 1}{\frac{1}{\operatorname{tg} x}} = 1$$

$$\frac{\operatorname{tg} x}{1-\operatorname{tg}^2 x} \cdot \frac{\frac{1-\operatorname{tg}^2 x}{\operatorname{tg}^2 x}}{\frac{1}{\operatorname{tg} x}} = 1$$

$$\frac{\operatorname{tg} x}{1-\operatorname{tg}^2 x} \cdot \frac{1-\operatorname{tg}^2 x}{\operatorname{tg} x} = 1$$

$$1 = 1$$

9)

$$\frac{\sin^2 x}{\sin x - \cos x} - \frac{\sin x + \cos x}{\operatorname{tg}^2 x - 1} = \sin x + \cos x$$

$$\frac{\sin^2 x}{\sin x - \cos x} - \frac{\sin x + \cos x}{\frac{\sin^2 x}{\cos^2 x} - 1} = \sin x + \cos x$$

$$\frac{\sin^2 x}{\sin x - \cos x} - \frac{\sin x + \cos x}{\frac{\sin^2 x - \cos^2 x}{\cos^2 x}} = \sin x + \cos x$$

$$\frac{\sin^2 x}{\sin x - \cos x} - \frac{\cos^2 x \cdot (\sin x + \cos x)}{(\sin x - \cos x)(\sin x + \cos x)} = \sin x + \cos x$$

$$\frac{\sin^2 x}{\sin x - \cos x} - \frac{\cos^2 x}{\sin x - \cos x} = \sin x + \cos x$$

$$\frac{\sin^2 x - \cos^2 x}{\sin x - \cos x} = \sin x + \cos x$$

$$\frac{(\sin x - \cos x)(\sin x + \cos x)}{\sin x - \cos x} = \sin x + \cos x$$

$$\sin x + \cos x = \sin x + \cos x$$

10)

$$\frac{\sin^3 x - \cos^3 x}{1 + \sin x \cos x} = \sin x - \cos x$$

$$\frac{(\sin x - \cos x)(\sin^2 x + \sin x \cos x + \cos^2 x)}{1 + \sin x \cos x} = \sin x - \cos x$$

$$\frac{(\sin x - \cos x)(1 + \sin x \cos x)}{1 + \sin x \cos x} = \sin x - \cos x$$

$$\sin x - \cos x = \sin x - \cos x$$