Zadatak 4. Dokaži sljedeće identitete:

1) $\dfrac{\sin x}{1+\operatorname{ctg} x}+\dfrac{\cos x}{1+\operatorname{tg} x}=\dfrac{1}{\sin x+\cos x}$;

2) $\dfrac{\sin x\cdot\cos x}{\cos^2 x-\sin^2 x}=\dfrac{\operatorname{tg} x}{1-\operatorname{tg}^2 x}$;

3) $\dfrac{1}{\sin^2 x-\cos^2 x}=\dfrac{1+\operatorname{ctg}^2 x}{1-\operatorname{ctg}^2 x}$;

4) $\left[\left(\dfrac{1+\cos x}{\sin x}\right)^2+1\right]:\dfrac{1+\cos x}{\sin^2 x}=2$;

5) $\dfrac{1+\operatorname{tg} x+\operatorname{tg}^2 x}{1+\operatorname{ctg} x+\operatorname{ctg}^2 x}=\operatorname{tg}^2 x$;

6) $\cos^2 x+2\sin^2 x+\sin^2 x\cdot\operatorname{tg}^2 x=\dfrac{1}{\cos^2 x}$;

7) $\cos^4 x+\sin^2 x\cdot\cos^2 x+\sin^2 x+\operatorname{tg}^2 x=\dfrac{1}{\cos^2 x}$;

8) $\dfrac{\sin x\cdot\operatorname{tg} x}{\sin x+\operatorname{tg} x}=\dfrac{\operatorname{tg} x-\sin x}{\sin x\cdot\operatorname{tg} x}$;

9) $\dfrac{\sin^2 x}{\cos x(1+\operatorname{tg} x)}-\dfrac{\cos^2 x}{\sin x(1+\operatorname{ctg} x)}=\sin x-\cos x$;

10) $\left(\dfrac{\sin x+\operatorname{tg} x}{\dfrac{1}{\sin x}+\operatorname{ctg} x}\right)^2=\dfrac{\sin^2 x+\operatorname{tg}^2 x}{\dfrac{1}{\sin^2 x}+\operatorname{ctg}^2 x}$.

Rješenje.

1)
$$\frac{\sin x}{1+\operatorname{ctg} x}+\frac{\cos x}{1+\operatorname{tg} x}=\frac{1}{\sin x+\cos x}$$
$$\frac{\sin x}{\dfrac{\sin x+\cos x}{\sin x}}+\frac{\cos x}{\dfrac{\cos x+\sin x}{\cos x}}=\frac{1}{\sin x+\cos x}$$
$$\frac{\sin^2 x}{\sin x+\cos x}+\frac{\cos^2 x}{\cos x+\sin x}=\frac{1}{\sin x+\cos x}$$
$$\frac{\sin^2 x+\cos^2 x}{\sin x+\cos x}=\frac{1}{\sin x+\cos x}$$
$$\frac{1}{\sin x+\cos x}=\frac{1}{\sin x+\cos x}$$

2)
$$\frac{\sin x\cdot\cos x}{\cos^2 x-\sin^2 x}=\frac{\operatorname{tg} x}{1-\operatorname{tg}^2 x}$$
$$\frac{\sin x\cdot\cos x}{\cos^2 x-\sin^2 x}=\frac{\dfrac{\sin x}{\cos x}}{\dfrac{\cos^2 x-\sin^2 x}{\cos^2 x}}$$
$$\frac{\sin x\cdot\cos x}{\cos^2 x-\sin^2 x}=\frac{\sin x\cdot\cos x}{\cos^2 x-\sin^2 x}$$

3)

$$\frac{1}{\sin^2 x - \cos^2 x} = \frac{1 + \operatorname{ctg}^2 x}{1 - \operatorname{ctg}^2 x}$$

$$\frac{1}{\sin^2 x - \cos^2 x} = \frac{1 + \frac{\cos^2 x}{\sin^2 x}}{1 - \frac{\cos^2 x}{\sin^2 x}}$$

$$\frac{1}{\sin^2 x - \cos^2 x} = \frac{\frac{\sin^2 x + \cos^2 x}{\sin^2 x}}{\frac{\sin^2 x - \cos^2 x}{\sin^2 x}}$$

$$\frac{1}{\sin^2 x - \cos^2 x} = \frac{1}{\sin^2 x - \cos^2 x}$$

4)

$$\left[\left(\frac{1 + \cos x}{\sin x}\right)^2 + 1\right] : \frac{1 + \cos x}{\sin^2 x} = 2$$

$$\frac{1 + \cos^2 x + 2\cos x + \sin^2 x}{\sin^2 x} \cdot \frac{\sin^2 x}{1 + \cos x} = 2$$

$$\frac{2 + 2\cos x}{\sin^2 x} \cdot \frac{\sin^2 x}{1 + \cos x} = 2$$

$$\frac{2(1 + \cos x)}{1 + \cos x} = 2$$

$$2 = 2$$

5)

$$\frac{1 + \operatorname{tg} x + \operatorname{tg}^2 x}{1 + \operatorname{ctg} x + \operatorname{ctg}^2 x} = \operatorname{tg}^2 x$$

$$\frac{1 + \operatorname{tg} x + \operatorname{tg}^2 x}{1 + \frac{1}{\operatorname{tg} x} + \frac{1}{\operatorname{tg}^2 x}} = \operatorname{tg}^2 x$$

$$\frac{1 + \operatorname{tg} x + \operatorname{tg}^2 x}{\frac{\operatorname{tg}^2 x + \operatorname{tg} x + 1}{\operatorname{tg}^2 x}} = \operatorname{tg}^2 x$$

$$\operatorname{tg}^2 x = \operatorname{tg}^2 x$$

6)

$$\begin{aligned}
\cos^2 x + 2\sin^2 x + \sin^2 x \cdot \operatorname{tg}^2 x &= \frac{1}{\cos^2 x}\\
\cos^2 x + \sin^2 x \cdot (2 + \operatorname{tg}^2 x) &= \frac{1}{\cos^2 x}\\
\cos^2 x + \sin^2 x \cdot \left(2 + \frac{\sin^2 x}{\cos^2 x}\right) &= \frac{1}{\cos^2 x}\\
\cos^2 x + \sin^2 x \cdot \frac{2\cos^2 x + \sin^2 x}{\cos^2 x} &= \frac{1}{\cos^2 x}\\
\cos^2 x + \sin^2 x \cdot \frac{\cos^2 x + \cos^2 x + \sin^2 x}{\cos^2 x} &= \frac{1}{\cos^2 x}\\
\cos^2 x + \frac{\sin^2 x \cdot (1 + \cos^2 x)}{\cos^2 x} &= \frac{1}{\cos^2 x}\\
\frac{\cos^4 x + \sin^2 x \cdot \cos^2 x + \sin^2 x}{\cos^2 x} &= \frac{1}{\cos^2 x}\\
\frac{\cos^2 x \cdot (\cos^2 x + \sin^2 x) + \sin^2 x}{\cos^2 x} &= \frac{1}{\cos^2 x}\\
\frac{\cos^2 x + \sin^2 x}{\cos^2 x} &= \frac{1}{\cos^2 x}\\
\frac{1}{\cos^2 x} &= \frac{1}{\cos^2 x}
\end{aligned}$$

7)

$$\begin{aligned}
\cos^4 x + \sin^2 x \cdot \cos^2 x + \sin^2 x + \operatorname{tg}^2 x &= \frac{1}{\cos^2 x}\\
\cos^2 x(\cos^2 x + \sin^2 x) + \sin^2 x + \frac{\sin^2 x}{\cos^2 x} &= \frac{1}{\cos^2 x}\\
\cos^2 x + \sin^2 x\left(1 + \frac{1}{\cos^2 x}\right) &= \frac{1}{\cos^2 x}\\
\cos^2 x + \sin^2 x \cdot \frac{\cos^2 x + 1}{\cos^2 x} &= \frac{1}{\cos^2 x}\\
\cos^2 x + \frac{\sin^2 x \cdot \cos^2 x + \sin^2 x}{\cos^2 x} &= \frac{1}{\cos^2 x}\\
\cos^2 x + \sin^2 x + \frac{\sin^2 x}{\cos^2 x} &= \frac{1}{\cos^2 x}\\
1 + \frac{\sin^2 x}{\cos^2 x} &= \frac{1}{\cos^2 x}\\
\frac{\cos^2 x + \sin^2 x}{\cos^2 x} &= \frac{1}{\cos^2 x}\\
\frac{1}{\cos^2 x} &= \frac{1}{\cos^2 x}
\end{aligned}$$

8)

$$\frac{\sin x \cdot \operatorname{tg} x}{\sin x + \operatorname{tg} x} = \frac{\operatorname{tg} x - \sin x}{\sin x \cdot \operatorname{tg} x}$$

$$\frac{\sin x \cdot \dfrac{\sin x}{\cos x}}{\sin x + \dfrac{\sin x}{\cos x}} = \frac{\dfrac{\sin x}{\cos x} - \sin x}{\sin x \cdot \dfrac{\sin x}{\cos x}}$$

$$\frac{\dfrac{\sin^2 x}{\cos x}}{\dfrac{\sin x \cdot \cos x + \sin x}{\cos x}} = \frac{\dfrac{\sin x - \sin x \cdot \cos x}{\cos x}}{\dfrac{\sin^2 x}{\cos x}}$$

$$\frac{\sin^2 x}{\sin x \cdot (\cos x + 1)} = \frac{\sin x \cdot (1 - \cos x)}{\sin^2 x}$$

$$\frac{\sin x}{\cos x + 1} = \frac{1 - \cos x}{\sin x} \quad \Big/ \sin x \cdot (\cos x + 1)$$

$$\sin^2 x = (1 - \cos x)(\cos x + 1)$$

$$\sin^2 x = 1 - \cos^2 x$$

$$\sin^2 x = \sin^2 x$$

9)

$$\frac{\sin^2 x}{\cos x(1 + \operatorname{tg} x)} - \frac{\cos^2 x}{\sin x(1 + \operatorname{ctg} x)} = \sin x - \cos x$$

$$\frac{\sin^2 x}{\cos x\left(1 + \dfrac{\sin x}{\cos x}\right)} - \frac{\cos^2 x}{\sin x\left(1 + \dfrac{\cos x}{\sin x}\right)} = \sin x - \cos x$$

$$\frac{\sin^2 x}{\cos x \cdot \dfrac{\cos x + \sin x}{\cos x}} - \frac{\cos^2 x}{\sin x \cdot \dfrac{\sin x + \cos x}{\sin x}} = \sin x - \cos x$$

$$\frac{\sin^2 x}{\cos x + \sin x} - \frac{\cos^2 x}{\sin x + \cos x} = \sin x - \cos x$$

$$\frac{\sin^2 x - \cos^2 x}{\sin x + \cos x} = \sin x - \cos x$$

$$\frac{(\sin x - \cos x)(\sin x + \cos x)}{\sin x + \cos x} = \sin x - \cos x$$

$$\sin x - \cos x = \sin x - \cos x$$

10)

$$\left(\frac{\sin x+\operatorname{tg} x}{\frac{1}{\sin x}+\operatorname{ctg} x}\right)^2=\frac{\sin^2 x+\operatorname{tg}^2 x}{\frac{1}{\sin^2 x}+\operatorname{ctg}^2 x}$$

$$\left(\frac{\sin x+\frac{\sin x}{\cos x}}{\frac{1}{\sin x}+\frac{\cos x}{\sin x}}\right)^2=\frac{\sin^2 x+\frac{\sin^2 x}{\cos^2 x}}{\frac{1}{\sin^2 x}+\frac{\cos^2 x}{\sin^2 x}}$$

$$\left(\frac{\frac{\sin x\cdot(\cos x+1)}{\cos x}}{\frac{1+\cos x}{\sin x}}\right)^2=\frac{\frac{\sin^2 x\cdot(\cos^2 x+1)}{\cos^2 x}}{\frac{1+\cos^2 x}{\sin^2 x}}$$

$$\left(\frac{\frac{\sin x}{\cos x}}{\frac{1}{\sin x}}\right)^2=\frac{\frac{\sin^2 x}{\cos^2 x}}{\frac{1}{\sin^2 x}}$$

$$\left(\frac{\sin^2 x}{\cos x}\right)^2=\frac{\sin^4 x}{\cos^2 x}$$

$$\frac{\sin^4 x}{\cos^2 x}=\frac{\sin^4 x}{\cos^2 x}$$