Zadatak 3. $\sin^6 x + \cos^6 x = a \cdot (\sin^4 x + \cos^4 x)\,.$

Rješenje.

$$\begin{aligned}
\sin^6 x + \cos^6 x &= a \cdot (\sin^4 x + \cos^4 x)\\
(\sin^2 x + \cos^2 x)(\sin^4 x - \sin^2 x\cos^2 x + \cos^4 x) &= a \cdot [(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x]\\
(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x - \sin^2 x\cos^2 x &= a \cdot [1 - 2\sin^2 x\cos^2 x]\\
1 - 3\sin^2 x\cos^2 x &= a \cdot (1 - 2\sin^2 x\cos^2 x)\\
1 - \frac{3}{4}\sin^2 2x &= a \cdot \left(1 - \frac{1}{2}\sin^2 2x\right)\\
1 - \frac{3}{4}\sin^2 2x &= a - \frac{a}{2}\sin^2 2x \quad / \cdot 4\\
4 - 3\sin^2 2x &= 4a - 2a\sin^2 2x\\
\sin^2 2x(2a-3) &= 4a - 4\\
\sin^2 2x &= \frac{4a-4}{2a-3}\\
\frac{1-\cos 4x}{2} &= \frac{4a-4}{2a-3}\\
1 - \cos 4x &= \frac{8a-8}{2a-3}\\
\cos 4x &= 1 - \frac{8a-8}{2a-3}\\
\cos 4x &= \frac{5-6a}{2a-3}\\
4x &= \arccos\frac{5-6a}{2a-3} + 2k\pi\\
x &= \frac{1}{4}\arccos\frac{5-6a}{2a-3} + k\cdot\frac{\pi}{2}, \quad k \in \mathbf{Z}
\end{aligned}$$

Uvjet na a:

$$|\cos 4x| \leqslant 1 \implies -1 \leqslant \frac{5-6a}{2a-3} \leqslant 1$$

1° $\quad a \neq \frac{3}{2}, \ a > \frac{3}{2}$

$$\frac{5-6a}{2a-3} \geqslant -1, \ 5-6a \geqslant 3-2a, \ 4a \leqslant 2, \ a \leqslant \frac{1}{2} \ \text{nije rješnje}$$

2° $\quad a \neq \frac{3}{2}, \ a < \frac{3}{2}$

$$\left.\begin{aligned}
&\frac{5-6a}{2a-3} \geqslant -1, \ 5-6a \leqslant 3-2a, \ 4a \geqslant 2, \ a \geqslant \frac{1}{2}\\
&\frac{5-6a}{2a-3} \leqslant 1, \ 5-6a \geqslant 3-2a, \ 8a \leqslant 8, \ a \leqslant 1
\end{aligned}\right\} \implies \frac{1}{2} \leqslant a \leqslant 1$$