

Zadatak 20. Ako je $\alpha + \beta + \gamma = \pi$, tada je

$$\sin^2 \frac{\alpha}{2} + \sin^2 \frac{\beta}{2} + \sin^2 \frac{\gamma}{2} \geq \frac{3}{4}.$$

Rješenje. Nejednakost možemo zapisati u obliku

$$\frac{1 - \cos \alpha}{2} + \frac{1 - \cos \beta}{2} + \frac{1 - \cos \gamma}{2} \geq \frac{3}{4},$$

$$\text{odnosno } \cos \alpha + \cos \beta + \cos \gamma \geq \frac{3}{2}.$$

No $\cos \gamma = \cos(\pi - (\alpha + \beta)) = -\cos(\alpha + \beta)$, onda imamo:

$$\begin{aligned} \cos \alpha + \cos \beta - \cos(\alpha + \beta) &\geq \frac{3}{2} = \\ 2 \cos \frac{\alpha+\beta}{2} \cos \frac{\alpha-\beta}{2} - 2 \cos^2 \frac{\alpha+\beta}{2} + 1 &\leq \frac{3}{2}. \end{aligned}$$

Konačno imamo:

$$\cos \frac{\alpha+\beta}{2} \cos \frac{\alpha-\beta}{2} - \cos^2 \frac{\alpha+\beta}{2} \leq \frac{1}{4},$$

a tu jednakost možemo zapisati u obliku

$$\frac{1}{4} \cos^2 \frac{\alpha-\beta}{2} - \left(\cos \frac{\alpha+\beta}{2} - \frac{1}{2} \cos \frac{\alpha-\beta}{2} \right)^2 \leq \frac{1}{4},$$

a to je očito točna nejednakost.