Zadatak 13. Dokaži sljedeće identitete:

1) $4\sin x\cdot\cos x-8\sin^3 x\cdot\cos x=\sin 4x$;

2) $\sin^4 x-6\sin^2 x\cdot\cos^2 x+\cos^4 x=\cos 4x$;

3) $\dfrac{\sin 3x}{\sin x}-\dfrac{\cos 3x}{\cos x}=2$;

4) $\dfrac{\sin 3x}{\cos x}+\dfrac{\cos 3x}{\sin x}=2\operatorname{ctg} 2x$;

5) $\operatorname{tg} x+\operatorname{ctg} x=\dfrac{2}{\sin 2x}$;

6) $\operatorname{ctg} x-\operatorname{tg} x=\dfrac{2}{\operatorname{tg} 2x}$;

7) $\dfrac{2\sin 2x-\sin 4x}{2\sin 2x+\sin 4x}=\operatorname{tg}^2 x$;

8) $\dfrac{2\sin 4x+\sin 8x}{2\sin 4x-\sin 8x}=\operatorname{ctg}^2 2x$;

9) $\dfrac{\operatorname{tg} 2x\cdot\operatorname{tg} x}{\operatorname{tg} 2x-\operatorname{tg} x}=\sin 2x$;

10) $\dfrac{1-\sin 2x}{1+\sin 2x}=\operatorname{ctg}^2\left(\dfrac{\pi}{4}+x\right)$;

11) $\dfrac{\cos 2x}{1+\sin 2x}=\dfrac{1-\operatorname{tg} x}{1+\operatorname{tg} x}$;

12) $\dfrac{\cos 4x+1}{\operatorname{ctg} x-\operatorname{tg} x}=\dfrac{1}{2}\sin 4x$;

13) $\dfrac{\sin 4x}{1+\cos 4x}\cdot\dfrac{\cos 2x}{1+\cos 2x}=\operatorname{ctg}\left(\dfrac{3\pi}{2}-x\right)$;

14) $\dfrac{\sin^4 x+2\sin x\cdot\cos x-\cos^4 x}{\operatorname{tg} 2x-1}=\cos 2x$;

15) $1+\operatorname{ctg}^4 x=\dfrac{1+\cos^2 2x}{2\sin^4 x}$.

Rješenje. **1)**

$$\begin{aligned}
4\sin x\cdot\cos x-8\sin^3 x\cdot\cos x&=\sin 4x\\
4\sin x\cdot\cos x(1-2\sin^2 x)&=\sin 4x\\
2\sin 2x(\sin^2 x+\cos^2 x-2\sin^2 x)&=\sin 4x\\
2\sin 2x(\cos^2 x-\sin^2 x)&=\sin 4x\\
2\sin 2x\cdot\cos 2x&=\sin 4x\\
\sin 4x&=\sin 4x
\end{aligned}$$

2)

$$\begin{aligned}
\sin^4 x-6\sin^2 x\cdot\cos^2 x+\cos^4 x&=\cos 4x\\
(\sin^2 x-\cos^2 x)^2-4\sin^2 x\cdot\cos^2 x&=\cos 4x\\
(-\cos 2x)^2-(\sin 2x)^2&=\cos 4x
\end{aligned}$$

$$\cos^2 2x - \sin^2 2x = \cos 4x$$
$$\cos 4x = \cos 4x$$

3)

$$\frac{\sin 3x}{\sin x} - \frac{\cos 3x}{\cos x} = 2$$
$$\frac{\sin 3x \cos x - \sin x \cos 3x}{\sin x \cos x} = 2$$
$$\frac{\sin(3x - x)}{\frac{\sin 2x}{2}} = 2$$
$$\frac{\sin 2x}{\frac{\sin 2x}{2}} = 2$$
$$2 = 2$$

4)

$$\frac{\sin 3x}{\cos x} + \frac{\cos 3x}{\sin x} = 2 \operatorname{ctg} 2x$$
$$\frac{\sin 3x \sin x + \cos x \cos 3x}{\sin x \cos x} = 2 \operatorname{ctg} 2x$$
$$\frac{\cos(3x - x)}{\frac{\sin 2x}{2}} = 2 \operatorname{ctg} 2x$$
$$2 \operatorname{ctg} 2x = 2 \operatorname{ctg} 2x$$

5)

$$\operatorname{tg} x + \operatorname{ctg} x = \frac{2}{\sin 2x}$$
$$\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = \frac{2}{\sin 2x}$$
$$\frac{\sin^2 x + \cos^2 x}{\cos x \sin x} = \frac{2}{\sin 2x}$$
$$\frac{1}{\frac{\sin 2x}{2}} = \frac{2}{\sin 2x}$$
$$\frac{2}{\sin 2x} = \frac{2}{\sin 2x}$$

6)

$$\operatorname{ctg} x - \operatorname{tg} x = \frac{2}{\operatorname{tg} 2x}$$
$$\frac{1}{\operatorname{tg} x} - \operatorname{tg} x = \frac{2}{\operatorname{tg} 2x}$$
$$\frac{1 - \operatorname{tg}^2 x}{\operatorname{tg} x} = \frac{2}{\operatorname{tg} 2x}$$

$$\frac{\frac{2}{\operatorname{tg} x}}{1-\operatorname{tg}^2 x}=\frac{2}{\operatorname{tg} 2x}$$
$$\frac{2}{\operatorname{tg} 2x}=\frac{2}{\operatorname{tg} 2x}$$

7)

$$\frac{2\sin 2x-\sin 4x}{2\sin 2x+\sin 4x}=\operatorname{tg}^2 x$$
$$\frac{2\sin 2x-2\sin 2x\cos 2x}{2\sin 2x+2\sin 2x\cos 2x}=\operatorname{tg}^2 x$$
$$\frac{2\sin 2x(1-\cos 2x)}{2\sin 2x(1+\cos 2x)}=\operatorname{tg}^2 x$$
$$\frac{1-\cos^2 x+\sin^2 x}{1+\cos^2 x-\sin^2 x}=\operatorname{tg}^2 x$$
$$\frac{\sin^2 x+\cos^2 x-\cos^2 x+\sin^2 x}{\sin^2 x+\cos^2 x+\cos^2 x-\sin^2 x}=\operatorname{tg}^2 x$$
$$\frac{2\sin^2 x}{2\cos^2 x}=\operatorname{tg}^2 x$$
$$\operatorname{tg}^2 x=\operatorname{tg}^2 x$$

8)

$$\frac{2\sin 4x+\sin 8x}{2\sin 4x-\sin 8x}=\operatorname{ctg}^2 2x$$
$$\frac{2\sin 4x+2\sin 4x\cos 4x}{2\sin 4x-2\sin 4x\cos 4x}=\operatorname{ctg}^2 2x$$
$$\frac{2\sin 4x(1+\cos 4x)}{2\sin 4x(1-\cos 4x)}=\operatorname{ctg}^2 2x$$
$$\frac{1+\cos^2 2x-\sin^2 2x}{1-\cos^2 2x+\sin^2 2x}=\operatorname{ctg}^2 2x$$
$$\frac{\sin^2 2x+\cos^2 2x+\cos^2 2x-\sin^2 2x}{\sin^2 2x+\cos^2 2x-\cos^2 2x+\sin^2 2x}=\operatorname{ctg}^2 2x$$
$$\frac{2\cos^2 2x}{2\sin^2 2x}=\operatorname{ctg}^2 2x$$
$$\operatorname{ctg}^2 2x=\operatorname{ctg}^2 2x$$

9)

$$\frac{\operatorname{tg} 2x\cdot\operatorname{tg} x}{\operatorname{tg} 2x-\operatorname{tg} x}=\sin 2x$$
$$\frac{\frac{2\operatorname{tg} x}{1-\operatorname{tg}^2 x}\cdot\operatorname{tg} x}{\frac{2\operatorname{tg} x}{1-\operatorname{tg}^2 x}-\operatorname{tg} x}=\sin 2x$$

$$\begin{aligned}
\frac{\frac{2\operatorname{tg}^2 x}{1-\operatorname{tg}^2 x}}{\operatorname{tg} x\cdot\left(\frac{2}{1-\operatorname{tg}^2 x}-1\right)} &= \sin 2x\\
\frac{\frac{2\operatorname{tg} x}{1-\operatorname{tg}^2 x}}{\frac{2-1+\operatorname{tg}^2 x}{1-\operatorname{tg}^2 x}} &= \sin 2x\\
\frac{2\operatorname{tg} x}{1+\operatorname{tg}^2 x} &= \sin 2x\\
\frac{2\frac{\sin x}{\cos x}}{1+\frac{\sin^2 x}{\cos^2 x}} &= \sin 2x\\
\frac{\frac{2\sin x}{\cos x}}{\frac{\cos^2 x+\sin^2 x}{\cos^2 x}} &= \sin 2x\\
2\sin x\cos x &= \sin 2x\\
\sin 2x &= \sin 2x
\end{aligned}$$

10)

$$\begin{aligned}
\frac{1-\sin 2x}{1+\sin 2x} &= \operatorname{ctg}^2\left(\frac{\pi}{4}+x\right)\\
\frac{1-2\sin x\cos x}{1+2\sin x\cos x} &= \left(\frac{\operatorname{ctg}\frac{\pi}{4}\operatorname{ctg} x-1}{\operatorname{ctg}\frac{\pi}{4}+\operatorname{ctg} x}\right)^2\\
\frac{\sin^2 x+\cos^2 x-2\sin x\cos x}{\sin^2 x+\cos^2 x+2\sin x\cos x} &= \left(\frac{\operatorname{ctg} x-1}{1+\operatorname{ctg} x}\right)^2\\
\frac{(\sin x-\cos x)^2}{(\sin x+\cos x)^2} &= \left(\frac{\frac{\cos x}{\sin x}-1}{1+\frac{\cos x}{\sin x}}\right)^2\\
\left(\frac{\sin x-\cos x}{\sin x+\cos x}\right)^2 &= \left(\frac{\frac{\cos x-\sin x}{\sin x}}{\frac{\sin x+\cos x}{\sin x}}\right)^2\\
\left(\frac{\sin x-\cos x}{\sin x+\cos x}\right)^2 &= \left(\frac{\sin x-\cos x}{\sin x+\cos x}\right)^2
\end{aligned}$$

11)

$$\frac{\cos 2x}{1+\sin 2x} = \frac{1-\operatorname{tg} x}{1+\operatorname{tg} x}$$
$$\frac{\cos^2 x - \sin^2 x}{\sin^2 x + \cos^2 x + 2\sin x\cos x} = \frac{1-\frac{\sin x}{\cos x}}{1+\frac{\sin x}{\cos x}}$$
$$\frac{(\cos x - \sin x)(\cos x + \sin x)}{(\cos x + \sin x)^2} = \frac{\frac{\cos x - \sin x}{\cos x}}{\frac{\cos x + \sin x}{\cos x}}$$
$$\frac{\cos x - \sin x}{\cos x + \sin x} = \frac{\cos x - \sin x}{\cos x + \sin x}$$

12)

$$\frac{\cos 4x + 1}{\operatorname{ctg} x - \operatorname{tg} x} = \frac{1}{2}\sin 4x$$
$$\frac{\cos^2 2x - \sin^2 2x + 1}{\frac{\cos x}{\sin x} - \frac{\sin x}{\cos x}} = \frac{1}{2}\cdot 2\sin 2x\cos 2x$$
$$\frac{\cos^2 2x - \sin^2 2x + \sin^2 2x + \cos^2 2x}{\frac{\cos^2 x - \sin^2 x}{\sin x\cos x}} = \sin 2x\cos 2x$$
$$\frac{2\cos^2 2x}{\frac{\cos 2x}{\frac{1}{2}\sin 2x}} = \sin 2x\cos 2x$$
$$\sin 2x\cos 2x = \sin 2x\cos 2x$$

13)

$$\frac{\sin 4x}{1+\cos 4x}\cdot\frac{\cos 2x}{1+\cos 2x} = \operatorname{ctg}\left(\frac{3\pi}{2} - x\right)$$
$$\frac{2\sin 2x\cos 2x}{1+\cos^2 2x - \sin^2 2x}\cdot\frac{\cos 2x}{1+\cos 2x} = \frac{\operatorname{ctg}\frac{3\pi}{2}\operatorname{ctg} x + 1}{\operatorname{ctg} x - \operatorname{ctg}\frac{3\pi}{2}}$$
$$\frac{2\sin 2x\cos 2x}{\sin^2 2x + \cos^2 2x + \cos^2 2x - \sin^2 2x}\cdot\frac{\cos 2x}{1+\cos 2x} = \frac{1}{\operatorname{ctg} x}$$
$$\frac{2\sin 2x\cos 2x}{2\cos^2 2x}\cdot\frac{\cos 2x}{1+\cos 2x} = \operatorname{tg} x$$
$$\frac{\sin 2x}{1+\cos 2x} = \operatorname{tg} x$$
$$\frac{2\sin x\cos x}{1+\cos^2 x - \sin^2 x} = \operatorname{tg} x$$
$$\frac{2\sin x\cos x}{\sin^2 x + \cos^2 x + \cos^2 x - \sin^2 x} = \operatorname{tg} x$$

$$\frac{2\sin x\cos x}{2\cos^2 x}=\operatorname{tg} x$$
$$\frac{\sin x}{\cos x}=\operatorname{tg} x$$
$$\operatorname{tg} x=\operatorname{tg} x$$

14)

$$\frac{\sin^4 x+2\sin x\cdot\cos x-\cos^4 x}{\operatorname{tg} 2x-1}=\cos 2x$$
$$\frac{(\sin^2 x-\cos^2 x)(\sin^2 x+\cos^2 x)+\sin 2x}{\frac{\sin 2x}{\cos 2x}-1}=\cos 2x$$
$$\frac{-\cos 2x+\sin 2x}{\frac{\sin 2x-\cos 2x}{\cos 2x}}=\cos 2x$$
$$\cos 2x=\cos 2x$$

15)

$$1+\operatorname{ctg}^4 x=\frac{1+\cos^2 2x}{2\sin^4 x}$$
$$1+\frac{\cos^4 x}{\sin^4 x}=\frac{1+\cos^2 2x}{2\sin^4 x}$$
$$\frac{\sin^4 x+\cos^4 x}{\sin^4 x}=\frac{1+\cos^2 2x}{2\sin^4 x}$$
$$\frac{(\sin^2 x+\cos^2 x)^2-2\sin^2 x\cos^2 x}{\sin^4 x}=\frac{1+\cos^2 2x}{2\sin^4 x}$$
$$\frac{1-\frac{1}{2}\sin^2 2x}{\sin^4 x}=\frac{1+\cos^2 2x}{2\sin^4 x}$$
$$\frac{2-\sin^2 2x}{2\sin^4 x}=\frac{1+\cos^2 2x}{2\sin^4 x}$$
$$\frac{1+\sin^2 2x+\cos^2 2x-\sin^2 2x}{2\sin^4 x}=\frac{1+\cos^2 2x}{2\sin^4 x}$$
$$\frac{1+\cos^2 2x}{2\sin^4 x}=\frac{1+\cos^2 2x}{2\sin^4 x}$$