Zadatak 14. Dokaži sljedeće identitete:

1) $\dfrac{\cos 3x}{\sin 2x} = \dfrac{1 - 4\sin^2 x}{2\sin x}$;

2) $\dfrac{\sin 2x}{\sin 3x} = \dfrac{2\cos x}{4\cos^2 x - 1}$;

3) $\dfrac{\cos x}{\operatorname{ctg}^2 \frac{x}{2} - \operatorname{tg}^2 \frac{x}{2}} = \dfrac{1}{4}\sin^2 x$;

4) $\dfrac{1 - 2\sin^2 \frac{x}{2}}{\operatorname{tg} \frac{x}{2} + \operatorname{ctg} \frac{x}{2}} = \dfrac{1}{4}\sin 2x$;

5) $\dfrac{1 + \cos 2x}{\sin 2x} \cdot \dfrac{1 + \cos x}{\cos x} = \operatorname{ctg} \dfrac{x}{2}$;

6) $\dfrac{1 + \sin\alpha - \cos\alpha}{1 + \sin\alpha + \cos\alpha} = \operatorname{tg} \dfrac{\alpha}{2}$;

7) $\dfrac{\sin\alpha + \sin \frac{\alpha}{2}}{1 + \cos\alpha + \cos \frac{\alpha}{2}} = \operatorname{tg} \dfrac{\alpha}{2}$;

8) $\dfrac{1 - \sin \frac{\alpha}{2} + \cos \frac{\alpha}{2}}{1 - \sin \frac{\alpha}{2} - \cos \frac{\alpha}{2}} = -\operatorname{ctg} \dfrac{\alpha}{4}$;

9) $\dfrac{\operatorname{tg}\left(\frac{\pi}{4} + \frac{\alpha}{2}\right) - \operatorname{tg}\left(\frac{\pi}{4} - \frac{\alpha}{2}\right)}{\operatorname{ctg}\left(\frac{\pi}{4} + \frac{\alpha}{2}\right) + \operatorname{ctg}\left(\frac{\pi}{4} - \frac{\alpha}{2}\right)} = \sin\alpha$;

10) $\dfrac{\sqrt{1 + \cos\alpha} + \sqrt{1 - \cos\alpha}}{\sqrt{1 + \cos\alpha} - \sqrt{1 - \cos\alpha}} = \operatorname{tg}\left(\dfrac{\pi}{4} - \dfrac{\alpha}{2}\right)$,
$\pi < \alpha < 2\pi$.

Rješenje. 1)

$$
\begin{aligned}
\frac{\cos 3x}{\sin 2x} &= \frac{1 - 4\sin^2 x}{2\sin x} \\
\frac{\cos(2x + x)}{2\sin x \cos x} &= \frac{1 - 4\sin^2 x}{2\sin x} \\
\frac{\cos 2x \cos x - \sin 2x \sin x}{2\sin x \cos x} &= \frac{1 - 4\sin^2 x}{2\sin x} \\
\frac{\cos 2x \cos x - 2\sin x \cos x \sin x}{2\sin x \cos x} &= \frac{1 - 4\sin^2 x}{2\sin x} \\
\frac{\cos x(\cos 2x - 2\sin^2 x)}{2\sin x \cos x} &= \frac{1 - 4\sin^2 x}{2\sin x} \\
\frac{\cos 2x - 2\sin^2 x}{2\sin x} &= \frac{1 - 4\sin^2 x}{2\sin x} \\
\frac{\cos^2 x - \sin^2 x - 2\sin^2 x}{2\sin x} &= \frac{1 - 4\sin^2 x}{2\sin x}
\end{aligned}
$$

$$\frac{1-\sin^2 x-3\sin^2 x}{2\sin x}=\frac{1-4\sin^2 x}{2\sin x}$$
$$\frac{1-4\sin^2 x}{2\sin x}=\frac{1-4\sin^2 x}{2\sin x}$$

2)

$$\frac{\sin 2x}{\sin 3x}=\frac{2\cos x}{4\cos^2 x-1}$$
$$\frac{2\sin x\cos x}{\sin(2x+x)}=\frac{2\cos x}{4\cos^2 x-1}$$
$$\frac{2\sin x\cos x}{\sin 2x\cos x+\cos 2x\sin x}=\frac{2\cos x}{4\cos^2 x-1}$$
$$\frac{2\sin x\cos x}{2\sin x\cos^2 x+\cos 2x\sin x}=\frac{2\cos x}{4\cos^2 x-1}$$
$$\frac{2\sin x\cos x}{\sin x(2\cos^2 x+\cos^2 x-\sin^2 x)}=\frac{2\cos x}{4\cos^2 x-1}$$
$$\frac{2\cos x}{3\cos^2 x-\sin^2 x}=\frac{2\cos x}{4\cos^2 x-1}$$
$$\frac{2\cos x}{3\cos^2 x-(1-\cos^2 x)}=\frac{2\cos x}{4\cos^2 x-1}$$
$$\frac{2\cos x}{4\cos^2 x-1}=\frac{2\cos x}{4\cos^2 x-1}$$

3)

$$\frac{\cos x}{\operatorname{ctg}^2\frac{x}{2}-\operatorname{tg}^2\frac{x}{2}}=\frac{1}{4}\sin^2 x$$
$$\frac{\cos x}{\dfrac{\cos^2\frac{x}{2}}{\sin^2\frac{x}{2}}-\dfrac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}}=\frac{1}{4}\sin^2 x$$
$$\frac{\cos x}{\dfrac{\cos^4\frac{x}{2}-\sin^4\frac{x}{2}}{\sin^2\frac{x}{2}\cos^2\frac{x}{2}}}=\frac{1}{4}\sin^2 x$$
$$\frac{\cos x}{\dfrac{\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)\left(\cos^2\frac{x}{2}+\sin^2\frac{x}{2}\right)}{\frac{1}{4}\sin^2 x}}=\frac{1}{4}\sin^2 x$$
$$\frac{\frac{1}{4}\cos x\cdot\sin^2 x}{\cos x}=\frac{1}{4}\sin^2 x$$
$$\frac{1}{4}\sin^2 x=\frac{1}{4}\sin^2 x$$

4)

$$\frac{1-2\sin^2\frac{x}{2}}{\operatorname{tg}\frac{x}{2}+\operatorname{ctg}\frac{x}{2}}=\frac{1}{4}\sin 2x$$

$$\frac{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}-2\sin^2\frac{x}{2}}{\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}+\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}}=\frac{1}{4}\sin 2x$$

$$\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\frac{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}}{\cos\frac{x}{2}\sin\frac{x}{2}}}=\frac{1}{4}\sin 2x$$

$$\frac{\cos x}{\frac{1}{\frac{1}{2}\sin x}}=\frac{1}{4}\sin 2x$$

$$\frac{1}{2}\sin x\cos x=\frac{1}{4}\sin 2x$$

$$\frac{1}{4}\sin 2x=\frac{1}{4}\sin 2x$$

5)

$$\frac{1+\cos 2x}{\sin 2x}\cdot\frac{1+\cos x}{\cos x}=\operatorname{ctg}\frac{x}{2}$$

$$\frac{(\sin^2 x+\cos^2 x)+(\cos^2 x-\sin^2 x)}{2\sin x\cos x}\cdot\frac{1+\cos x}{\cos x}=\operatorname{ctg}\frac{x}{2}$$

$$\frac{2\cos^2 x}{2\sin x\cos x}\cdot\frac{1+\cos x}{\cos x}=\operatorname{ctg}\frac{x}{2}$$

$$\frac{1+\cos x}{\sin x}=\operatorname{ctg}\frac{x}{2}$$

$$\frac{1+\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{2\sin\frac{x}{2}}=\operatorname{ctg}\frac{x}{2}$$

$$\frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}=\operatorname{ctg}\frac{x}{2}$$

$$\operatorname{ctg}\frac{x}{2}=\operatorname{ctg}\frac{x}{2}$$

6)

$$\frac{1+\sin\alpha-\cos\alpha}{1+\sin\alpha+\cos\alpha}=\operatorname{tg}\frac{\alpha}{2}$$
$$\frac{1+\sin\alpha-\cos^2\frac{\alpha}{2}+\sin^2\frac{\alpha}{2}}{1+\sin\alpha+\cos^2\frac{\alpha}{2}-\sin^2\frac{\alpha}{2}}=\operatorname{tg}\frac{\alpha}{2}$$
$$\frac{\sin\alpha+2\sin^2\frac{\alpha}{2}}{\sin\alpha+2\cos^2\frac{\alpha}{2}}=\operatorname{tg}\frac{\alpha}{2}$$
$$\frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}+2\sin^2\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}+2\cos^2\frac{\alpha}{2}}=\operatorname{tg}\frac{\alpha}{2}$$
$$\frac{2\sin\frac{\alpha}{2}\left(\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}\right)}{2\cos\frac{\alpha}{2}\left(\sin\frac{\alpha}{2}+\cos\frac{\alpha}{2}\right)}=\operatorname{tg}\frac{\alpha}{2}$$
$$\operatorname{tg}\frac{\alpha}{2}=\operatorname{tg}\frac{\alpha}{2}$$

7)

$$\frac{\sin\alpha+\sin\frac{\alpha}{2}}{1+\cos\alpha+\cos\frac{\alpha}{2}}=\operatorname{tg}\frac{\alpha}{2}$$
$$\frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}{1+\cos^2\frac{\alpha}{2}-\sin^2\frac{\alpha}{2}+\cos\frac{\alpha}{2}}=\operatorname{tg}\frac{\alpha}{2}$$
$$\frac{\sin\frac{\alpha}{2}\left(2\cos\frac{\alpha}{2}+1\right)}{2\cos^2\frac{\alpha}{2}+\cos\frac{\alpha}{2}}=\operatorname{tg}\frac{\alpha}{2}$$
$$\frac{\sin\frac{\alpha}{2}\left(2\cos\frac{\alpha}{2}+1\right)}{\cos\frac{\alpha}{2}\left(2\cos\frac{\alpha}{2}+1\right)}=\operatorname{tg}\frac{\alpha}{2}$$
$$\operatorname{tg}\frac{\alpha}{2}=\operatorname{tg}\frac{\alpha}{2}$$

8)

$$\frac{1-\sin\frac{\alpha}{2}+\cos\frac{\alpha}{2}}{1-\sin\frac{\alpha}{2}-\cos\frac{\alpha}{2}}=-\operatorname{ctg}\frac{\alpha}{4}$$
$$\frac{1-2\sin\frac{\alpha}{4}\cos\frac{\alpha}{4}+\cos^2\frac{\alpha}{4}-\sin^2\frac{\alpha}{4}}{1-2\sin\frac{\alpha}{4}\cos\frac{\alpha}{4}-\cos^2\frac{\alpha}{4}+\sin^2\frac{\alpha}{4}}=-\operatorname{ctg}\frac{\alpha}{4}$$

$$\frac{-2\sin\frac{\alpha}{4}\cos\frac{\alpha}{4}+2\cos^2\frac{\alpha}{4}}{-2\sin\frac{\alpha}{4}\cos\frac{\alpha}{4}+2\sin^2\frac{\alpha}{4}}=-\operatorname{ctg}\frac{\alpha}{4}$$

$$\frac{-2\cos\frac{\alpha}{4}\left(\sin\frac{\alpha}{4}-\cos\frac{\alpha}{4}\right)}{2\sin\frac{\alpha}{4}\left(\sin\frac{\alpha}{4}-\cos\frac{\alpha}{4}\right)}=-\operatorname{ctg}\frac{\alpha}{4}$$

$$-\operatorname{ctg}\frac{\alpha}{4}=-\operatorname{ctg}\frac{\alpha}{4}$$

9)

$$\frac{\operatorname{tg}\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)-\operatorname{tg}\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}{\operatorname{ctg}\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)+\operatorname{ctg}\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}=\sin\alpha$$

$$\frac{\dfrac{\operatorname{tg}\frac{\pi}{4}+\operatorname{tg}\frac{\alpha}{2}}{1-\operatorname{tg}\frac{\pi}{4}\operatorname{tg}\frac{\alpha}{2}}-\dfrac{\operatorname{tg}\frac{\pi}{4}-\operatorname{tg}\frac{\alpha}{2}}{1+\operatorname{tg}\frac{\pi}{4}\operatorname{tg}\frac{\alpha}{2}}}{\dfrac{\operatorname{ctg}\frac{\pi}{4}\operatorname{ctg}\frac{\alpha}{2}-1}{\operatorname{ctg}\frac{\pi}{4}+\operatorname{ctg}\frac{\alpha}{2}}+\dfrac{\operatorname{ctg}\frac{\pi}{4}\operatorname{ctg}\frac{\alpha}{2}+1}{\operatorname{ctg}\frac{\pi}{4}-\operatorname{ctg}\frac{\alpha}{2}}}=\sin\alpha$$

$$\frac{\dfrac{1+\operatorname{tg}\frac{\alpha}{2}}{1-\operatorname{tg}\frac{\alpha}{2}}-\dfrac{1-\operatorname{tg}\frac{\alpha}{2}}{1+\operatorname{tg}\frac{\alpha}{2}}}{\dfrac{\operatorname{ctg}\frac{\alpha}{2}-1}{1+\operatorname{ctg}\frac{\alpha}{2}}+\dfrac{\operatorname{ctg}\frac{\alpha}{2}+1}{1-\operatorname{ctg}\frac{\alpha}{2}}}=\sin\alpha$$

$$\frac{\dfrac{\left(1+\operatorname{tg}\frac{\alpha}{2}\right)^2-\left(1-\operatorname{tg}\frac{\alpha}{2}\right)^2}{\left(1-\operatorname{tg}\frac{\alpha}{2}\right)\left(1+\operatorname{tg}\frac{\alpha}{2}\right)}}{\dfrac{\left(\operatorname{ctg}\frac{\alpha}{2}-1\right)^2+\left(\operatorname{ctg}\frac{\alpha}{2}+1\right)^2}{\left(\operatorname{ctg}\frac{\alpha}{2}+1\right)\left(\operatorname{ctg}\frac{\alpha}{2}-1\right)}}=\sin\alpha$$

$$\frac{\dfrac{1+2\operatorname{tg}\frac{\alpha}{2}+\operatorname{tg}^2\frac{\alpha}{2}-1+2\operatorname{tg}\frac{\alpha}{2}-\operatorname{tg}^2\frac{\alpha}{2}}{1-\operatorname{tg}^2\frac{\alpha}{2}}}{\dfrac{\operatorname{ctg}^2\frac{\alpha}{2}-2\operatorname{ctg}\frac{\alpha}{2}+1+\operatorname{ctg}^2\frac{\alpha}{2}-2\operatorname{ctg}\frac{\alpha}{2}+1}{\operatorname{ctg}^2\frac{\alpha}{2}-1}}=\sin\alpha$$

$$\frac{\dfrac{4\operatorname{tg}\dfrac{\alpha}{2}}{1-\operatorname{tg}^2\dfrac{\alpha}{2}}}{\dfrac{2\operatorname{ctg}^2\dfrac{\alpha}{2}+2}{\operatorname{ctg}^2\dfrac{\alpha}{2}-1}}=\sin\alpha$$

$$\frac{2\operatorname{tg}\dfrac{\alpha}{2}\left(\operatorname{ctg}^2\dfrac{\alpha}{2}-1\right)}{\left(1-\operatorname{tg}^2\dfrac{\alpha}{2}\right)\left(\operatorname{ctg}^2\dfrac{\alpha}{2}+1\right)}=\sin\alpha$$

$$\frac{2\dfrac{\sin\dfrac{\alpha}{2}}{\cos\dfrac{\alpha}{2}}\cdot\dfrac{\cos^2\dfrac{\alpha}{2}-\sin^2\dfrac{\alpha}{2}}{\sin^2\dfrac{\alpha}{2}}}{\dfrac{\cos^2\dfrac{\alpha}{2}-\sin^2\dfrac{\alpha}{2}}{\cos^2\dfrac{\alpha}{2}}\cdot\dfrac{\cos^2\dfrac{\alpha}{2}+\sin^2\dfrac{\alpha}{2}}{\sin^2\dfrac{\alpha}{2}}}=\sin\alpha$$

$$\frac{\dfrac{2}{\cos\dfrac{\alpha}{2}}\cdot\dfrac{\cos\alpha}{\sin\dfrac{\alpha}{2}}}{\dfrac{\cos\alpha}{\cos^2\dfrac{\alpha}{2}}\cdot\dfrac{1}{\sin^2\dfrac{\alpha}{2}}}=\sin\alpha$$

$$2\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}=\sin\alpha$$

$$\sin\alpha=\sin\alpha$$

10)

$$\frac{\sqrt{1+\cos\alpha}+\sqrt{1-\cos\alpha}}{\sqrt{1+\cos\alpha}-\sqrt{1-\cos\alpha}}=\operatorname{tg}\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\quad \pi<\alpha<2\pi$$

$$\frac{\sqrt{2}\cos\dfrac{\alpha}{2}+\sqrt{2}\sin\dfrac{\alpha}{2}}{\sqrt{2}\cos\dfrac{\alpha}{2}-\sqrt{2}\sin\dfrac{\alpha}{2}}=\frac{\operatorname{tg}\dfrac{\pi}{4}+\operatorname{tg}\dfrac{\alpha}{2}}{1-\operatorname{tg}\dfrac{\pi}{4}\operatorname{tg}\dfrac{\alpha}{2}}$$

$$\frac{\cos\dfrac{\alpha}{2}+\sin\dfrac{\alpha}{2}}{\cos\dfrac{\alpha}{2}-\sin\dfrac{\alpha}{2}}=\frac{1+\operatorname{tg}\dfrac{\alpha}{2}}{1-\operatorname{tg}\dfrac{\alpha}{2}}$$

$$\frac{\cos\dfrac{\alpha}{2}+\sin\dfrac{\alpha}{2}}{\cos\dfrac{\alpha}{2}-\sin\dfrac{\alpha}{2}}=\frac{\dfrac{\cos\dfrac{\alpha}{2}+\sin\dfrac{\alpha}{2}}{\cos\dfrac{\alpha}{2}}}{\dfrac{\cos\dfrac{\alpha}{2}-\sin\dfrac{\alpha}{2}}{\cos\dfrac{\alpha}{2}}}$$

$$\frac{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}=\frac{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}$$