

Zadatak 18. Ako je $\alpha + \beta + \gamma = \pi$, dokaži da je onda

$$\sin^2 \frac{\alpha}{2} + \sin^2 \frac{\beta}{2} + \sin^2 \frac{\gamma}{2} \geq \frac{3}{4}.$$

Rješenje. Danu nejednakost možemo zapisati kao $\frac{1 - \cos \alpha}{2} + \frac{1 - \cos \beta}{2} + \frac{1 - \cos \gamma}{2} \geq \frac{3}{4}$, odnosno $\cos \alpha + \cos \beta + \cos \gamma \leq \frac{3}{2}$. No, $\alpha + \beta + \gamma = \pi$ te je $\cos \gamma = -\cos(\alpha + \beta)$. Također je $\cos \gamma = 2 \cos^2 \frac{\gamma}{2} - 1$ te $\cos \alpha + \cos \beta + \cos \gamma = \cos \alpha + \cos \beta - \cos(\alpha + \beta) = 2 \cos \frac{\alpha + \beta}{2} \cdot \cos \frac{\alpha - \beta}{2}$. Tako imamo: $\cos \alpha + \cos \beta + \cos \gamma = \cos \alpha + \cos \beta - \cos(\alpha + \beta) = 2 \cos \frac{\alpha + \beta}{2} \cdot \cos \frac{\alpha - \beta}{2} - 2 \cos^2 \frac{\alpha + \beta}{2} + 1$, a ovu jednakost možemo zapisati u obliku $\cos \frac{\alpha + \beta}{2} \cdot \cos \frac{\alpha - \beta}{2} - \cos^2 \frac{\alpha + \beta}{2} \leq \frac{1}{4}$, odnosno $\frac{1}{4} \cos^2 \frac{\alpha - \beta}{2} - (\cos \frac{\alpha + \beta}{2} - \frac{1}{2} \cos \frac{\alpha - \beta}{2})^2 \leq \frac{1}{4}$.