Zadatak 2. Matematičkom indukcijom dokaži da za sve $n \in \mathbf{N}$ vrijedi:

1) $2+7+15+\ldots+\frac{1}{2}n(3n+1)=\frac{1}{2}n(n+1)^2$;

2) $-1+3-5+\ldots+(-1)^n(2n-1)=(-1)^n \cdot n$;

3) $3+6+12+\ldots+3\cdot 2^{n-1}=3(2^n-1)$;

4) $2+16+56+\ldots+(3n-2)\cdot 2^n=10+(3n-5)\cdot 2^{n+1}$.

Rješenje. **1)** Dokazujemo indukcijom:

$$2+7+15+\ldots+\frac{1}{2}n(3n+1)=\frac{1}{2}n(n+1)^2.$$

Baza. $n=1$:

$$2=\frac{1}{2}\cdot 1\cdot(1+1)^2.$$

Pretpostavka. Za $n=k$ vrijedi:

$$2+7+15+\ldots+\frac{1}{2}k(3k+1)=\frac{1}{2}k(k+1)^2.$$

Korak. Računamo lijevu stranu za $n=k+1$ i koristimo pretpostavku indukcije:

$$\begin{aligned}
2+7+15+\ldots+&\frac{1}{2}k(3k+1)+\frac{1}{2}(k+1)[3(k+1)+1]\\
&=\frac{1}{2}k(k+1)^2+\frac{1}{2}(k+1)[3(k+1)+1]\\
&=\frac{1}{2}(k+1)\Big[k(k+1)+3(k+1)+1\Big]\\
&=\frac{1}{2}(k+1)\Big[k^2+k+3k+3+1\Big]\\
&=\frac{1}{2}(k+1)\Big[k^2+4k+4\Big]\\
&=\frac{1}{2}(k+1)(k+2)^2\\
&=\frac{1}{2}(k+1)[(k+1)+1]^2.
\end{aligned}$$

Tvrdnja je dokazana.

2) Dokazujemo indukcijom:

$$-1+3-5+\ldots+(-1)^n(2n-1)=(-1)^n\cdot n.$$

Baza. $n=1$:

$$-1=(-1)^1\cdot 1.$$

Pretpostavka. Za $n=k$ vrijedi:

$$-1+3-5+\ldots+(-1)^k(2k-1)=(-1)^k\cdot k.$$

Korak. Računamo lijevu stranu za $n = k + 1$ i koristimo pretpostavku indukcije:

$$\begin{aligned}-1 + 3 - 5 + \ldots + (-1)^k(2k-1) + (-1)^{k+1}[2(k+1)-1] &= (-1)^k \cdot k + (-1)^{k+1}[2(k+1)-1]\\ &= (-1)^k \cdot k + (-1)^{k+1}(2k+1)\\ &= (-1)^k[k-(2k+1)]\\ &= (-1)^k[-k-1]\\ &= -(-1)^k[k+1]\\ &= (-1)^{k+1}(k+1).\end{aligned}$$

Tvrdnja je dokazana.

3) Dokazujemo indukcijom:

$$3 + 6 + 12 + \ldots + 3 \cdot 2^{n-1} = 3(2^n - 1).$$

Baza. $n = 1$:

$$3 = 3(2^1 - 1).$$

Pretpostavka. Za $n = k$ vrijedi:

$$3 + 6 + 12 + \ldots + 3 \cdot 2^{k-1} = 3(2^k - 1).$$

Korak. Računamo lijevu stranu za $n = k + 1$ i koristimo pretpostavku indukcije:

$$\begin{aligned}3 + 6 + 12 + \ldots + 3 \cdot 2^{k-1} + 3 \cdot 2^k &= 3(2^k - 1) + 3 \cdot 2^k\\ &= 3 \cdot 2^k - 3 + 3 \cdot 2^k\\ &= 6 \cdot 2^k - 3\\ &= 3 \cdot 2^{k+1} - 3\\ &= 3(2^{k+1} - 1).\end{aligned}$$

Tvrdnja je dokazana.

4) Dokazujemo indukcijom:

$$2 + 16 + 56 + \ldots + (3n-2) \cdot 2^n = 10 + (3n-5) \cdot 2^{n+1}.$$

Baza. $n = 1$:

$$2 = 10 + (3 \cdot 1 - 5) \cdot 2^{1+1}.$$

Pretpostavka. Za $n = k$ vrijedi:

$$2 + 16 + 56 + \ldots + (3k-2) \cdot 2^k = 10 + (3k-5) \cdot 2^{k+1}.$$

Korak. Računamo lijevu stranu za $n = k + 1$ i koristimo pretpostavku indukcije:

$$\begin{aligned}
2 + 16 + 56 + \ldots + (3k-2) \cdot 2^k + [3(k+1)-2] \cdot 2^{k+1} \\
&= 10 + (3k-5) \cdot 2^{k+1} + [3(k+1)-2] \cdot 2^{k+1} \\
&= 10 + (3k-5) \cdot 2^{k+1} + [3k+1] \cdot 2^{k+1} \\
&= 10 + 2^{k+1}[3k-5+3k+1] \\
&= 10 + 2^{k+1}[6k-4] \\
&= 10 + 2^{k+2}[3k-2] \\
&= 10 + [3(k+1)-5] \cdot 2^{(k+1)+1}.
\end{aligned}$$

Tvrdnja je dokazana.