Zadatak 6. Dokaži matematičkom indukcijom:

1) $\frac{1}{1\cdot 5}+\frac{1}{5\cdot 9}+\ldots+\frac{1}{(4n-3)(4n+1)}$
$=\frac{n}{4n+1}$;

2) $\frac{1}{1\cdot 3}+\frac{2^2}{3\cdot 5}+\frac{3^2}{5\cdot 7}+\ldots+\frac{n^2}{(2n-1)(2n+1)}$
$=\frac{n(n+1)}{2(2n+1)}$;

3) $\frac{3}{4}+\frac{5}{36}+\ldots+\frac{2n+1}{n^2(n+1)^2}=1-\frac{1}{(n+1)^2}$;

4) $\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\ldots\left(1-\frac{1}{(n+1)^2}\right)$
$=\frac{n+2}{2(n+1)}$;

5) $\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\ldots+\frac{1}{n^2+3n+2}=\frac{n}{2(n+2)}$;

6) $\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\ldots+\frac{n}{2^n}=2-\frac{n+2}{2^n}$.

Rješenje. **1)** Zbroj s lijeve strane ima samo jedan član ako je $n=1$. *Baza*, $n=1$:

$$\frac{1}{5}=\frac{1}{4\cdot 1+1}.$$

Pretpostavimo da tvrdnja vrijedi za broj n:

$$\frac{1}{1\cdot 5}+\frac{1}{5\cdot 9}+\ldots+\frac{1}{(4n-3)(4n+1)}=\frac{n}{4n+1}.$$

Dokažimo da ona onda vrijedi i za broj $n+1$:

$$\begin{aligned}\frac{1}{1\cdot 5}+\frac{1}{5\cdot 9}+\ldots+\frac{1}{(4n-3)(4n+1)}+\frac{1}{(4n+1)(4n+5)}&=\frac{n}{4n+1}+\frac{1}{(4n+1)(4n+5)}\\&=\frac{n(4n+5)+1}{(4n+1)(4n+5)}\\&=\frac{4n^2+5n+1}{(4n+1)(4n+5)}\\&=\frac{(4n+1)(n+1)}{(4n+1)(4n+5)}\\&=\frac{n+1}{4n+5}.\end{aligned}$$

Dakle, tvrdnja vrijedi za svaki n.

2) *Baza*, $n=1$:

$$\frac{1}{3}=\frac{1^2}{(2\cdot 1-1)(2\cdot 1+1)}.$$

Pretpostavimo da tvrdnja vrijedi za broj n:

$$\frac{1}{1\cdot 3}+\frac{2^2}{3\cdot 5}+\frac{3^2}{5\cdot 7}+\ldots+\frac{n^2}{(2n-1)(2n+1)}=\frac{n(n+1)}{2(2n+1)}.$$

Dokažimo da ona onda vrijedi i za broj $n+1$:

$$\begin{aligned}
\frac{1}{1\cdot 3}+\frac{2^2}{3\cdot 5}+\frac{3^2}{5\cdot 7}+\ldots+\frac{n^2}{(2n-1)(2n+1)}+\frac{(n+1)^2}{(2n+1)(2n+3)}
&=\frac{n(n+1)}{2(2n+1)}+\frac{(n+1)^2}{(2n+1)(2n+3)}\\
&=\frac{n(n+1)(2n+3)+2(n+1)^2}{2(2n+1)(2n+3)}\\
&=\frac{(n+1)[n(2n+3)+2(n+1)]}{2(2n+1)(2n+3)}\\
&=\frac{(n+1)[2n^2+3n+2n+2]}{2(2n+1)(2n+3)}\\
&=\frac{(n+1)[2n^2+5n+2]}{2(2n+1)(2n+3)}\\
&=\frac{(n+1)(2n+1)(n+2)}{2(2n+1)(2n+3)}\\
&=\frac{(n+1)(n+2)}{2(2n+3)}.
\end{aligned}$$

Dakle, tvrdnja vrijedi za svaki n.

3) *Baza*, $n=1$:

$$\frac{3}{4}=1-\frac{1}{(1+1)^2}.$$

Pretpostavimo da tvrdnja vrijedi za broj n:

$$\frac{3}{4}+\frac{5}{36}+\ldots+\frac{2n+1}{n^2(n+1)^2}=1-\frac{1}{(n+1)^2}.$$

Dokažimo da ona onda vrijedi i za broj $n+1$:

$$\begin{aligned}
\frac{3}{4}+\frac{5}{36}+\ldots+\frac{2n+1}{n^2(n+1)^2}+\frac{2n+3}{(n+1)^2(n+2)^2}
&=1-\frac{1}{(n+1)^2}+\frac{2n+3}{(n+1)^2(n+2)^2}\\
&=1-\frac{(n+2)^2-(2n+3)}{(n+1)^2(n+2)^2}\\
&=1-\frac{n^2+4n+4-2n-3}{(n+1)^2(n+2)^2}\\
&=1-\frac{n^2+2n+1}{(n+1)^2(n+2)^2}\\
&=1-\frac{(n+1)^2}{(n+1)^2(n+2)^2}\\
&=1-\frac{1}{(n+2)^2}.
\end{aligned}$$

Dakle, tvrdnja vrijedi za svaki n.

4) *Baza*, $n = 1$:

$$1 - \frac{1}{4} = \frac{1+2}{2(1+1)};$$
$$\frac{3}{4} = \frac{3}{4}.$$

Pretpostavimo da tvrdnja vrijedi za broj n:

$$\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{9}\right)\ldots\left(1 - \frac{1}{(n+1)^2}\right) = \frac{n+2}{2(n+1)}.$$

Dokažimo da ona onda vrijedi i za broj $n+1$:

$$\begin{aligned}\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{9}\right)\ldots\left(1 - \frac{1}{(n+1)^2}\right)\left(1 - \frac{1}{(n+2)^2}\right)\\ &= \left(\frac{n+2}{2(n+1)}\right)\left(1 - \frac{1}{(n+2)^2}\right)\\ &= \frac{n+2}{2(n+1)} \cdot \frac{(n+2)^2 - 1}{(n+2)^2}\\ &= \frac{(n+2)[(n+2)^2 - 1]}{2(n+1)(n+2)^2}\\ &= \frac{(n+2)[n^2 + 4n + 3]}{2(n+1)(n+2)^2}\\ &= \frac{(n+2)(n+1)(n+3)}{2(n+1)(n+2)^2}\\ &= \frac{n+3}{2(n+2)}.\end{aligned}$$

Dakle, tvrdnja vrijedi za svaki n.

5) *Baza*, $n = 1$:

$$\frac{1}{6} = \frac{1}{2(1+2)}.$$

Izraz $n^2 + 3n + 2$ u nazivniku napišimo kao $(n+1)(n+2)$. Pretpostavimo da tvrdnja vrijedi za broj n:

$$\frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \ldots + \frac{1}{(n+1)(n+2)} = \frac{n}{2(n+2)}.$$

Dokažimo da ona onda vrijedi i za broj $n+1$:

$$\begin{aligned}\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\ldots+\frac{1}{(n+1)(n+2)}+\frac{1}{(n+2)(n+3)} &= \frac{n}{2(n+2)}+\frac{1}{(n+2)(n+3)}\\ &= \frac{n(n+3)+2}{2(n+2)(n+3)}\\ &= \frac{n^2+3n+2}{2(n+2)(n+3)}\\ &= \frac{(n+1)(n+2)}{2(n+2)(n+3)}\\ &= \frac{n+1}{2(n+3)}.\end{aligned}$$

Dakle, tvrdnja vrijedi za svaki n.

6) *Baza*, $n=1$:

$$\frac{1}{2}=2-\frac{1+2}{2^1};$$
$$\frac{1}{2}=\frac{1}{2}.$$

Pretpostavimo da tvrdnja vrijedi za broj n:

$$\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\ldots+\frac{n}{2^n}=2-\frac{n+2}{2^n}.$$

Dokažimo da ona onda vrijedi i za broj $n+1$:

$$\begin{aligned}\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\ldots+\frac{n}{2^n}+\frac{n+1}{2^{n+1}} &= 2-\frac{n+2}{2^n}+\frac{n+1}{2^{n+1}}\\ &= 2-\frac{2(n+2)-(n+1)}{2^{n+1}}\\ &= 2-\frac{n+3}{2^{n+1}}.\end{aligned}$$

Dakle, tvrdnja vrijedi za svaki n.