Zadatak 18. Dokaži matematičkom indukcijom:

1) $\sin x + \sin 2x + \sin 3x + \ldots + \sin nx$
$$= \frac{\sin \frac{n+1}{2}x}{\sin \frac{x}{2}} \cdot \sin \frac{nx}{2};$$

2) $\cos x + \cos 2x + \cos 3x + \ldots + \cos nx$
$$= \frac{\cos \frac{n+1}{2}x}{\sin \frac{x}{2}} \cdot \sin \frac{nx}{2};$$

3) $\sin x + \sin 3x + \sin 5x + \ldots + \sin(2n-1)x$
$$= \frac{\sin^2 nx}{\sin x};$$

4) $\cos x + \cos 3x + \cos 5x + \ldots + \cos(2n-1)x$
$$= \frac{\sin 2nx}{2\sin x}.$$

Rješenje. **1)** Za $n = 1$ tvrdnja očito vrijedi (zbroj ima samo jedan pribrojnik). Pretpostavimo da tvrdnja vrijedi za prirodni broj $n = k$, tj. da je

$$\sin x + \sin 2x + \ldots + \sin kx = \frac{\sin \frac{k+1}{2}x}{\sin \frac{x}{2}} \cdot \sin \frac{kx}{2}.$$

Dokažimo da tvrdnja onda vrijedi i za sljedeći broj $k+1$. Imamo redom:

$$\begin{aligned}
\sin x &+ \sin 2x + \ldots + \sin kx + \sin(k+1)x \\
&= \frac{\sin \frac{k+1}{2}x}{\sin \frac{x}{2}} \cdot \sin \frac{kx}{2} + \sin(k+1)x \\
&= \frac{\sin \frac{k+1}{2}x \cdot \sin \frac{kx}{2} + 2\sin \frac{k+1}{2}x \cdot \cos \frac{k+1}{2}x \cdot \sin \frac{x}{2}}{\sin \frac{x}{2}} \\
&= \frac{\sin \frac{k+1}{2}x}{\sin \frac{x}{2}} \left(\sin \frac{kx}{2} + 2\sin \frac{x}{2} \cdot \cos \frac{k+1}{2}x \right) \\
&= \frac{\sin \frac{k+1}{2}x}{\sin \frac{x}{2}} \left(\sin \frac{kx}{2} + \sin \frac{(k+2)x}{2} - \sin \frac{kx}{2} \right) \\
&= \frac{\sin \frac{k+2}{2}x}{\sin \frac{x}{2}} \cdot \sin \frac{(k+1)x}{2},
\end{aligned}$$

a to je i trebalo dobiti.

2) Za $n=1$ tvrdnja očito vrijedi (suma ima samo jedan pribrojnik). Pretpostavimo da tvrdnja vrijedi za prirodni broj $n=k$, tj. da je

$$\cos x+\cos 2x+\ldots+\cos kx=\frac{\cos\frac{k+1}{2}x}{\sin\frac{x}{2}}\cdot\sin\frac{kx}{2}.$$

Dokažimo da tvrdnja onda vrijedi i za sljedeći broj $k+1$. Imamo redom:

$$\begin{aligned}\cos x+&\cos 2x+\ldots+\cos kx+\cos(k+1)x\\ &=\frac{\cos\frac{k+1}{2}x}{\sin\frac{x}{2}}\cdot\sin\frac{kx}{2}+\cos(k+1)x\\ &=\frac{\cos\frac{k+1}{2}x\cdot\sin\frac{kx}{2}+\cos(k+1)x\sin\frac{x}{2}}{\sin\frac{x}{2}}.\end{aligned}$$

Transformirajmo brojnik:

$$\begin{aligned}&\cos\frac{k+1}{2}x\cdot\sin\frac{kx}{2}+\cos(k+1)x\sin\frac{x}{2}\\ &=\cos\frac{k+1}{2}x\cdot\sin\frac{(k+1)x-x}{2}+\cos(k+1)x\sin\frac{x}{2}\\ &=\cos\frac{k+1}{2}x\cdot\left[\sin\frac{k+1}{2}x\cos\frac{x}{2}-\sin\frac{x}{2}\cos\frac{k+1}{2}x\right]+\left[\cos^2\frac{k+1}{2}x-\sin^2\frac{k+1}{2}x\right]\sin\frac{x}{2}\\ &=\cos\frac{k+1}{2}x\cdot\sin\frac{k+1}{2}x\cos\frac{x}{2}-\sin\frac{x}{2}\cos^2\frac{k+1}{2}x+\cos^2\frac{k+1}{2}x\sin\frac{x}{2}-\sin^2\frac{k+1}{2}x\sin\frac{x}{2}\\ &=\sin\frac{k+1}{2}x\left[\cos\frac{k+1}{2}x\cos\frac{x}{2}-\sin\frac{k+1}{2}x\sin\frac{x}{2}\right]\\ &=\sin\frac{k+1}{2}x\cos\frac{k+2}{2}x.\end{aligned}$$

3) Za $n=1$ dobivamo istinitu jednakost

$$\sin x=\frac{\sin^2 x}{\sin x}=\sin x.$$

Pretpostavimo da tvrdnja vrijedi za broj n. Onda za $n+1$ dobivamo

$$\begin{aligned}\sum_{k=1}^{n+1}\sin(2k-1)x &= \frac{\sin^2 nx}{\sin x}+\sin(2n+1)x\\ &= \frac{\sin^2 nx+\sin(2n+1)x\sin x}{\sin x}\\ &= \frac{\frac{1}{2}(1-\cos 2nx)+\frac{1}{2}(\cos 2nx-\cos(2n+2)x)}{\sin x}\\ &= \frac{\frac{1}{2}(1-\cos 2nx+\cos 2nx-\cos(2n+2)x)}{\sin x}\\ &= \frac{\frac{1}{2}(1-\cos 2(n+1)x)}{\sin x}\\ &= \frac{\sin^2(n+1)x}{\sin x}.\end{aligned}$$

4) Za $n=1$ dobivamo istinitu jednakost

$$\cos x=\frac{\sin 2x}{2\sin x}=\frac{2\sin x\cos x}{2\sin x}=\cos x.$$

Pretpostavimo da tvrdnja vrijedi za broj n. Onda za $n+1$ dobivamo

$$\begin{aligned}\sum_{k=1}^{n+1}\cos(2k-1)x &= \frac{\sin 2nx}{2\sin x}+\cos(2n+1)x\\ &= \frac{\sin 2nx+2\sin x\cos(2n+1)x}{2\sin x}\\ &= \frac{\sin 2nx+\sin(-2nx)+\sin(2n+2)x}{2\sin x}\\ &= \frac{\sin 2nx-\sin 2nx+\sin 2(n+1)x}{2\sin x}\\ &= \frac{\sin 2(n+1)x}{2\sin x}.\end{aligned}$$