Zadatak 19. Matematičkom indukcijom dokaži:

1) $\cos x \cdot \cos 2x \cdot \cos 4x \cdot \ldots \cdot \cos 2^n x = \dfrac{\sin 2^{n+1}x}{2^{n+1}\sin x}$;

2) $\cos\dfrac{x}{2} \cdot \cos\dfrac{x}{2^2} \cdot \cos\dfrac{x}{2^3} \cdot \ldots \cdot \cos\dfrac{x}{2^n} = \dfrac{\sin x}{2^n \sin\dfrac{x}{2^n}}$;

3) $\dfrac{1}{2} + \cos x + \cos 2x + \ldots + \cos nx = \dfrac{\sin\dfrac{2n+1}{2}x}{2\sin\dfrac{x}{2}}$;

4) $\dfrac{1}{2}\operatorname{tg}\dfrac{x}{2} + \dfrac{1}{2^2}\operatorname{tg}\dfrac{x}{2^2} + \ldots + \dfrac{1}{2^n}\operatorname{tg}\dfrac{x}{2^n}$
$= \dfrac{1}{2^n}\operatorname{ctg}\dfrac{x}{2^n} - \operatorname{ctg} x$.

Rješenje. **1)** *Baza.* Za $n = 1$ imamo $\cos x \cdot \cos 2x = \dfrac{\sin 4x}{4\sin x}$, što je točno. Naime,

$$\frac{\sin 4x}{4\sin x} = \frac{2\sin 2x\cos 2x}{4\sin x} = \frac{4\sin x\cos x\cos 2x}{4\sin x} = \cos x\cos 2x.$$

Pretpostavka. Pretpostavimo da je tvrdnja istinita za $n = k$, da vrijedi

$$\cos x \cdot \cos 2x \cdot \ldots \cdot \cos 2^k x = \frac{\sin 2^{k+1}x}{2^{k+1}\sin x}.$$

Korak. Izračunajmo sada umnožak za $k+1$ uz uvažavanje ove pretpostavke

$$\cos x \cdot \cos 2x \cdot \ldots \cdot \cos 2^k x \cdot \cos 2^{k+1}x = \frac{\sin 2^{k+1}x}{2^{k+1}\sin x} \cdot \cos 2^{k+1}x$$
$$= \frac{2\sin 2^{k+1}x\cos 2^{k+1}x}{2^{k+2}\sin x} = \frac{\sin 2^{k+2}x}{2^{k+2}\sin x}.$$

Očito, pretpostavka ima za posljedicu valjanost tvrdnje za prvi sljedeći prirodni broj, pa iskazana tvrdnja vrijedi za svaki prirodni broj n.

2) *Baza.* Za $n = 1$ računamo desnu stranu:

$$\frac{\sin x}{2\sin\frac{x}{2}} = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin\frac{x}{2}} = \cos\frac{x}{2}.$$

Pretpostavka. Pretpostavimo da je istinita jednakost

$$\prod_{k=1}^{n}\cos\frac{x}{2^k} = \frac{\sin x}{2^n\sin\frac{x}{2^n}}.$$

Korak. $n := n+1$

$$\prod_{k=1}^{n+1} \cos\frac{x}{2^k} = \frac{\sin x}{2^n \sin\frac{x}{2^n}} \cdot \cos\frac{x}{2^{n+1}}$$

$$= \frac{\sin x}{2^n \cdot 2\sin\frac{x}{2^{n+1}} \cos\frac{x}{2^{n+1}}} \cdot \cos\frac{x}{2^{n+1}}$$

$$= \frac{\sin x}{2^{n+1} \sin\frac{x}{2^{n+1}}}.$$

3) Dokazujemo identitet

$$\frac{1}{2} + \sum_{k=1}^{n} \cos kx = \frac{\sin\frac{2n+1}{2}x}{2\sin\frac{x}{2}}.$$

Baza $n = 1$

$$\frac{1}{2} + \cos x = \frac{\sin\frac{3}{2}x}{2\sin\frac{x}{2}} = \frac{\sin x \cos\frac{x}{2} + \sin\frac{x}{2}\cos x}{2\sin\frac{x}{2}}$$

$$= \frac{2\sin\frac{x}{2}\cos^2\frac{x}{2} + \sin\frac{x}{2}\cos^2\frac{x}{2} - \sin^3\frac{x}{2}}{2\sin\frac{x}{2}}$$

$$= \frac{\sin\frac{x}{2}\left(2\cos^2\frac{x}{2} + \cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right)}{2\sin\frac{x}{2}}$$

$$= \frac{4\cos^2\frac{x}{2} - 1}{2} = \frac{4\left(\frac{1+\cos x}{2}\right) - 1}{2}$$

$$= \frac{2 + 2\cos x - 1}{2} = \frac{1 + 2\cos x}{2}$$

$$= \frac{1}{2} + \cos x;$$

Pretpostavka. Pretpostavimo da je identitet istinit za prirodni broj n.
Korak $n := n+1$

$$\frac{1}{2} + \sum_{k=1}^{n+1} \cos kx = \frac{\sin\frac{2n+1}{2}x}{2\sin\frac{x}{2}} + \cos(n+1)x$$

$$= \frac{\sin\frac{2n+1}{2}x + 2\sin\frac{x}{2}\cos(n+1)x}{2\sin\frac{x}{2}}$$

$$= \frac{1}{2\sin\frac{x}{2}}\left[\sin\frac{2n+1}{2}x + \sin\left(-\frac{2n+1}{2}\right)x + \sin\frac{2n+3}{2}x\right]$$

$$= \frac{1}{2\sin\frac{x}{2}}\left[\sin\frac{2n+1}{2}x - \sin\frac{2n+1}{2}x + \sin\frac{2n+3}{2}x\right]$$

$$= \frac{\sin\frac{2n+3}{2}x}{2\sin\frac{x}{2}}.$$

4) Dokazujemo identitet

$$\sum_{k=1}^{n}\frac{1}{2^k}\operatorname{tg}\frac{x}{2^k} = \frac{1}{2^n}\operatorname{ctg}\frac{x}{2^n} - \operatorname{ctg}x.$$

Baza $n = 1$

$$\frac{1}{2}\operatorname{tg}\frac{x}{2} = \frac{1}{2}\operatorname{ctg}\frac{x}{2} - \operatorname{ctg}x = \frac{1}{2\operatorname{tg}\frac{x}{2}} - \frac{\operatorname{ctg}^2\frac{x}{2} - 1}{2\operatorname{ctg}\frac{x}{2}}$$

$$= \frac{1}{2\operatorname{tg}\frac{x}{2}} - \frac{\frac{1}{\operatorname{tg}^2\frac{x}{2}} - 1}{\frac{2}{\operatorname{tg}\frac{x}{2}}} = \frac{1}{2\operatorname{tg}\frac{x}{2}} - \frac{\frac{1-\operatorname{tg}^2\frac{x}{2}}{\operatorname{tg}^2\frac{x}{2}}}{\frac{2}{\operatorname{tg}\frac{x}{2}}}$$

$$= \frac{1}{2\operatorname{tg}\frac{x}{2}} - \frac{1-\operatorname{tg}^2\frac{x}{2}}{2\operatorname{tg}\frac{x}{2}} = \frac{1-1+\operatorname{tg}^2\frac{x}{2}}{2\operatorname{tg}\frac{x}{2}} = \frac{1}{2}\operatorname{tg}\frac{x}{2}$$

Pretpostavka. Pretpostavimo da je identitet istinit za prirodni broj n.

Korak $n := n + 1$

$$\sum_{k=1}^{n+1}\frac{1}{2^k}\operatorname{tg}\frac{x}{2^k} = \frac{1}{2^n}\operatorname{ctg}\frac{x}{2^n} - \operatorname{ctg}x + \frac{1}{2^{n+1}}\operatorname{tg}\frac{x}{2^{n+1}}$$

$$= \frac{1}{2^n}\cdot\frac{\operatorname{ctg}^2\frac{x}{2^{n+1}} - 1}{2\operatorname{ctg}\frac{x}{2^{n+1}}} - \operatorname{ctg}x + \frac{1}{2^{n+1}}\cdot\frac{1}{\operatorname{ctg}\frac{x}{2^{n+1}}}$$

$$= \frac{\operatorname{ctg}^2\frac{x}{2^{n+1}} - 1 + 1}{2^{n+1}\operatorname{ctg}\frac{x}{2^{n+1}}} - \operatorname{ctg}x$$

$$= \frac{1}{2^{n+1}}\operatorname{ctg}\frac{x}{2^n+1} - \operatorname{ctg}x.$$