Zadatak 6. Odredi skup svih točaka z u kompleksnoj ravnini za koje je $\left(\frac{z-1}{z}\right)^2$ realan.

Rješenje.

$$z = a + bi$$

$$w = \left(\frac{z-1}{z}\right)^2 \in \mathbf{R} \iff \operatorname{Im} w = 0$$

$$w = \left(\frac{z-1}{z}\right)^2 = \left(1 + \frac{1}{z}\right)^2$$

$$\frac{1}{z} = \frac{1}{a+bi} = \frac{1}{a+bi} \cdot \frac{a-bi}{a-bi} = \frac{a-bi}{a^2+b^2}$$

$$w = \left(1 - \frac{a-bi}{a^2+b^2}\right)^2 = \left(\frac{a^2+b^2-a+bi}{a^2+b^2}\right)^2 = \left[\left(1 - \frac{a}{a^2+b^2}\right) + \frac{b}{a^2+b^2} i\right]^2$$

$$= \left(1 - \frac{a}{a^2+b^2}\right)^2 + 2\left(1 - \frac{a}{a^2+b^2}\right)\frac{b}{a^2+b^2} i - \left(\frac{b}{a^2+b^2}\right)^2$$

$$\operatorname{Im} w = 2\frac{(a^2+b^2-a)b}{a^2+b^2} = 0 \iff (a^2+b^2-a)b = 0$$

$$\iff b = 0 \text{ ili } a^2+b^2-a = 0 \iff b = 0 \text{ ili } \left(a - \frac{1}{2}\right)^2 + b^2 = \frac{1}{4}.$$