Zadatak 8.   Odredi skup svih kompleksnih brojeva z određenih uvjetima

1) $\operatorname{Im} z^2 = 2$;   **2)** $|z - i| + |z + i| = 4$;

3) $|z - 1| = |z - i|$;   **4)** $\operatorname{Re}(1 + z) = |z|$;

5) $|z-3|+|z+3|=8$;   **6)** $|z - 1| - |z + 1| = 1$;

7) $\operatorname{Re} \dfrac{z - 2}{z - 2i} = 0$;   **8)** $\operatorname{Im} \dfrac{z - 2}{z - 2i} = 0$;

9) $\operatorname{Re} \dfrac{z - z_1}{z - z_2} = 0$;   **10)** $\operatorname{Im} \dfrac{z - z_1}{z - z_1} = 0$.

Rješenje.   **1)** $\operatorname{Im} z^2 = 2$,

$$z^2 = (a + bi)^2 = a^2 - b^2 + 2abi,$$

$$\operatorname{Im} z^2 = 2 \iff 2ab = 2 \iff \boxed{ab = 1} \text{ hiperbola.}$$

2) $|z - i| + |z + i| = 4$,

$$|a + (b - 1)i| + |a + (b + 1)i| = 4$$

$$\sqrt{a^2 + (b - 1)^2} + \sqrt{a^2 + (b + 1)^2} = 4$$

$$\sqrt{a^2 + (b - 1)^2} = 4 - \sqrt{a^2 + (b + 1)^2} \; /^2$$

$$a^2 + b^2 - 2b + 1 = 16 + a^2 + b^2 + 2b + 1 - 8\sqrt{a^2 + (b + 1)^2}$$

$$-4b - 16 = -8\sqrt{a^2 + (b + 1)^2} \; / : (-4)$$

$$b + 4 = 2\sqrt{a^2 + (b + 1)^2} \; /^2$$

$$b^2 + 8b + 16 = 4a^2 + 4b^2 + 8b + 4$$

$$\iff \boxed{4a^2 + 3b^2 = 12} \text{ elipsa (žarišta } F(0, \pm 1)).$$

3) $|z - 1| = |z - i|$,

$$|z - 1| = |z - i|$$

$$|(a - 1) + bi| = |a + (b - 1)i| \; /^2$$

$$(a - 1)^2 + b^2 = a^2 + (b - 1)^2$$

$$a^2 - 2a + 1 + b^2 = a^2 + b^2 - 2b + 1$$

$$-2a = -2b \; / : (-2)$$

$$\boxed{b = a} \text{ pravac.}$$

4) $\operatorname{Re}(1 + z) = |z|$,

$$\operatorname{Re}(1 + a + bi) = \sqrt{a^2 + b^2}$$

$$a + 1 = \sqrt{a^2 + b^2} \; /^2$$

$$a^2 + 2a + 1 = a^2 + b^2$$

$$b^2 = 2a + 1$$

$$\boxed{b^2 = 2\left(a + \frac{1}{2}\right)} \text{ parabola.}$$

5) $|z - 3| + |z + 3| = 8$,

$$\sqrt{(a-3)^2 + b^2} = 8 - \sqrt{(a+3)^2 + b^2} \ /^2$$

$$a^2 - 6a + 9 + b^2 = 64 + a^2 + 6a + 9 + b^2 - 16\sqrt{(a+3)^2 + b^2}$$

$$-64 - 12a = -16\sqrt{(a+3)^2 + b^2} \ / : (-4)$$

$$16 + 3a = 4\sqrt{(a+3)^2 + b^2} \ /^2$$

$$256 + 96a + 9a^2 = 16a^2 + 96a + 144 + 16b^2$$

$$\boxed{7a^2 + 16b^2 = 112} \ \text{elipsa (žarišta } F(\pm 3, 0)\text{).}$$

6) $|z - 1| - |z + 1| = 1$,

$$\sqrt{(a-1)^2 + b^2} = 1 + \sqrt{(a+1)^2 + b^2} \ /^2$$

$$a^2 - 2a + 1 + b^2 = 1 + a^2 + 2a + 1 + b^2 + 2\sqrt{(a+1)^2 + b^2}$$

$$-4a - 1 = 2\sqrt{(a+1)^2 + b^2} \ /^2$$

$$16a^2 + 8a + 1 = 4a^2 + 8a + 4 + 4b^2$$

$$\boxed{12a^2 - 4b^2 = 3} \ \text{hiperbola.}$$

7) $\operatorname{Re} \dfrac{z - 2}{z - 2i} = 0$,

$$w = \frac{z-2}{z-2i} = \frac{(a-2)+bi}{a+(b-2)i} \cdot \frac{a-(b-2)i}{a-(b-2)i} = \frac{a(a-2)+abi-(a-2)(b-2)i+b(b-2)}{a^2+(b-2)^2}$$

$$\operatorname{Re} w = \frac{a(a-2)+b(b-2)}{a^2+(b-2)^2} = 0 \iff a^2 - 2a + b^2 - 2b = 0$$

$$\iff \boxed{(a-1)^2 + (b-1)^2 = 2} \ \text{kružnica.}$$

8) $\operatorname{Im} \dfrac{z - 2}{z - 2i} = 0$,

$$w = \frac{z-2}{z-2i} = \frac{a(a-2)+b(b-2)+[ab-(a-2)(b-2)]i}{a^2+(b-2)^2}$$

$$\operatorname{Im} w = \frac{ab-(a-2)(b-2)}{a^2+(b-2)^2} = 0 \iff ab - (a-2)(b-2) = 0$$

$$\iff ab - (ab - 2a - 2b + 4) = 0$$

$$\iff ab - ab + 2a + 2b - 4 = 0 \iff 2b = -2a + 4 \ / : 2$$

$$\iff \boxed{b = -a + 2} \ \text{pravac.}$$

9) $\operatorname{Re} \dfrac{z - z_1}{z - z_2} = 0$ **10)** $\operatorname{Im} \dfrac{z - z_1}{z - z_2} = 0$

$$w = \frac{z-z_1}{z-z_2} = \frac{a+bi-a_1-b_1i}{a+bi-a_2-b_2i} = \frac{(a-a_1)+(b-b_1)i}{(a-a_2)+(b-b_2)i} \cdot \frac{(a-a_2)-(b-b_2)i}{(a-a_2)-(b-b_2)i}$$

$$= \frac{[(a-a_1)(a-a_2)+(b-b_1)(b-b_2)] + [(a-a_2)(b-b_1)+(a-a_1)(b-b_2)]i}{(a-a_2)^2 + (b-b_2)^2}$$

$$\operatorname{Re} w = 0 \iff (a - a_1)(a - a_2) + (b - b_1)(b - b_2) = 0$$

$$\iff a^2 - (a_1 + a_2)a + a_1 a_2 + b^2 - (b_1 + b_2)b + b_1 b_2 = 0$$

$$\iff \left(a - \frac{a_1 + a_2}{2}\right)^2 + \left(b - \frac{b_1 + b_2}{2}\right)^2 = \left(\frac{a_1 + a_2}{2}\right)^2 + \left(\frac{b_1 + b_2}{2}\right)^2 - a_1 a_2 - b_1 b_2$$

$$\iff \boxed{\left(a - \frac{a_1 + a_2}{2}\right)^2 + \left(b - \frac{b_1 + b_2}{2}\right)^2 = \left(\frac{a_1 + a_2}{2}\right)^2 + \left(\frac{b_1 + b_2}{2}\right)^2} \quad \text{kružnica.}$$

$$\operatorname{Im} w = 0 \iff (a - a_2)(b - b_1) - (a - a_1)(b - b_2) = 0$$

$$\iff ab - a_2 b - a b_1 + a_2 b_1 - (ab - a_1 b - a b_2 + a_1 b_2) = 0$$

$$\iff ab - a_2 b - a b_1 + a_2 b_1 - ab + a_1 b + a b_2 - a_1 b_2 = 0$$

$$\iff (a_1 - a_2)b = (b_1 - b_2)a + a_1 b_2 - a_2 b_1$$

$$\iff \boxed{b = \frac{b_1 - b_2}{a_1 - a_2}a + \frac{a_1 b_2 - a_2 b_1}{a_1 - a_2}} \quad \text{pravac kroz točke } (a_1, b_1) \text{ i } (a_2, b_2).$$