Zadatak 9. Dokaži da je za svaki prirodni broj n ispunjena jednakost:

1) $k! + k \cdot k! + (k+1) \cdot (k+1)! + \ldots + n \cdot n! = (n+1)!$;

2) $(n+1) \cdot (n+2) \cdot \ldots \cdot (n+n) = 2^n \cdot 1 \cdot 3 \cdot \ldots \cdot (2n-1)$;

3) $\binom{n}{1} + 2\binom{n}{2} + \ldots + n\binom{n}{n} = n \cdot 2^{n-1}$;

4) $\binom{n}{0} + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \ldots + \frac{1}{n+1}\binom{n}{n} = \frac{2^{n+1}-1}{n+1}$.

Rješenje. **2)** Dokaz provodimo matematičkom indukcijom.

Baza.

Za $n = 1$ dobije se $(1+1) = 2^1 \cdot (2 \cdot 1 - 1)$, tj. $2 = 2$. Tvrdnja vrijedi za $n = 1$;

Pretpostavka.

Pretpostavimo da vrijedi $(n+1) \cdot (n+2) \cdot \ldots \cdot (n+n) = 2^n \cdot 1 \cdot 3 \cdot \ldots \cdot (2n-1)$;

Korak.

Dokažimo tvrdnju za $n+1$ raspisujući izraz s lijeve strane i koristeći pretpostavku.

$$\begin{aligned}
&[(n+1)+1] \cdot [(n+1)+2] \cdot \ldots \cdot [(n+1)+n] \cdot [(n+1)+(n+1)] \\
&\quad = (n+2) \cdot (n+3) \cdot \ldots \cdot (n+n+2) \\
&\quad = (n+1) \cdot (n+2) \cdot (n+3) \cdot \ldots \cdot (n+n) \cdot \frac{(n+n+1)(n+n+2)}{n+1} \\
&\quad = 2^n \cdot 1 \cdot 3 \cdot \ldots \cdot (2n-1) \cdot \frac{4n^2+6n+2}{n+1} \\
&\quad = 2^n \cdot 1 \cdot 3 \cdot \ldots \cdot (2n-1) \cdot \frac{2(n+1)(2n+1)}{n+1} \\
&\quad = 2^{n+1} \cdot 1 \cdot 3 \cdot \ldots \cdot (2n-1) \cdot (2n+1).
\end{aligned}$$

3) Dokaz provodimo matematičkom indukcijom.

Baza.

Za $n = 1$ dobije se $\binom{1}{1} = 1 \cdot 2^{1-1}$, tj. $1 = 1$. Tvrdnja vrijedi za $n = 1$;

Pretpostavka.

Pretpostavimo da vrijedi $\binom{n}{1} + 2\binom{n}{2} + \ldots + n\binom{n}{n} = n \cdot 2^{n-1}$;

Korak.

Dokažimo tvrdnju za $n+1$ raspisujući izraz s lijeve strane i koristeći pretpostavku.

$$\begin{aligned}
\binom{n+1}{1} + 2\binom{n+1}{2} + \ldots + n\binom{n+1}{n} + (n+1)\binom{n+1}{n+1} \\
&= \binom{n}{0} + \binom{n}{1} + 2\binom{n}{1} + 2\binom{n}{2} + \ldots + n\binom{n}{n-1} + n\binom{n}{n} + n + 1 \\
&= \binom{n}{0} + 3\binom{n}{1} + 5\binom{n}{2} + \ldots + [2(n-1)+1]\binom{n}{n-1} + n\binom{n}{n} + n\binom{n}{n} + 1\binom{n}{n} \\
&= \binom{n}{0} + 3\binom{n}{1} + 5\binom{n}{2} + \ldots + [2(n-1)+1]\binom{n}{n-1} + (2n+1)\binom{n}{n} \\
&= \sum_{k=0}^{n}(2k+1)\binom{n}{k} = 2\sum_{k=0}^{n}k\binom{n}{k} + \sum_{k=0}^{n}\binom{n}{k} = 2\sum_{k=1}^{n}k\binom{n}{k} + \sum_{k=0}^{n}\binom{n}{k} \\
&= 2 \cdot \left(n \cdot 2^{n-1}\right) + 2^n = n \cdot 2^n + 2^n = (n+1) \cdot 2^n.
\end{aligned}$$