Zadatak 11. Deriviraj sljedeće funkcije:

1) $f(x)=x\sin x$;

2) $f(x)=x-\sin x\cos x$;

3) $f(x)=x\sin x\cos x$;

4) $f(x)=\operatorname{tg}x-\operatorname{ctg}x$;

5) $f(x)=\dfrac{1-\operatorname{tg}x}{1+\operatorname{tg}x}$;

6) $f(x)=\operatorname{tg}x+\dfrac{1}{\cos x}$;

7) $f(x)=\dfrac{x+\sin x}{x-\sin x}$;

8) $f(x)=\dfrac{x+\operatorname{tg}x}{x-\operatorname{tg}x}$;

9) $f(x)=\dfrac{\sin x-\cos x}{\sin x+\cos x}$;

10) $f(x)=\dfrac{\operatorname{tg}x-\operatorname{ctg}x}{\operatorname{tg}x+\operatorname{ctg}x}$.

Rješenje. **1)** $f'(x)=(x\sin x)'=x'\sin x+x(\sin x)'=x^{1-1}\sin x+x\cdot\cos x=\sin x+x\cos x$;

2) $f'(x)=(x-\sin x\cos x)'=x'-(\sin x\cos x)'=1-[(\sin x)'\cos x+\sin x(\cos x)'=1-[\cos x\cdot\cos x+\sin x\cdot(-\sin x)==1-(\cos^2x-\sin^2x)=1-\cos 2x=2\sin^2x$;

3) $f'(x)=(x\sin x\cos x)'=x'\cdot(\sin x\cos x)+x(\sin x)'\cos x+x\sin x(\cos x)'=\sin x\cos x+x\cos^2x-x\sin^2x=\sin x\cos x+x(\cos^2x-\sin^2x)=\dfrac{1}{2}\sin 2x+x\cos 2x$;

4) $f'(x)=(\operatorname{tg}x-\operatorname{ctg}x)'=(\operatorname{tg}x)'-(\operatorname{ctg}x)'=\dfrac{1}{\cos^2x}+\dfrac{1}{\sin^2x}=\dfrac{\sin^2x+\cos^2x}{\sin^2x\cos^2x}=\dfrac{1}{\frac{1}{4}\sin^2 2x}=\dfrac{4}{\sin^2 2x}$;

5) $f'(x)=\left(\dfrac{1-\operatorname{tg}x}{1+\operatorname{tg}x}\right)'=\left(\dfrac{1-\frac{\sin x}{\cos x}}{1+\frac{\sin x}{\cos x}}\right)=\left(\dfrac{\cos x-\sin x}{\cos x+\sin x}\right)$

$$=\frac{(\cos x-\sin x)'(\cos x+\sin x)-(\cos x-\sin x)(\cos x+\sin x)'}{(\cos x+\sin x)^2}$$

$$=\frac{[(\cos x)'-(\sin x)'](\cos x+\sin x)-(\cos x-\sin x)[(\cos x)'+(\sin x)']}{(\cos x+\sin x)^2}$$

$$=\frac{(-\sin x-\cos x)(\cos x+\sin x)-(\cos x-\sin x)(-\sin x+\cos x)}{(\cos x+\sin x)^2}$$

$$=\frac{-(\sin x+\cos x)^2+(\cos x-\sin x)^2}{\cos^2x+2\cos x\sin x+\sin^2x}$$

$$=\frac{-(\cos^2x+2\sin x\cos x+\sin^2x)-(\sin^2x-2\sin x\cos x+\cos^2x)}{1+2\sin x\cos x}$$

$$=\frac{-1-2\sin x\cos x-1+2\sin x\cos x}{1+\sin 2x}=-\frac{2}{1+\sin 2x};$$

6) $f'(x)=\left(\operatorname{tg}x+\frac{1}{\cos x}\right)'=\left(\frac{\sin x}{\cos x}+\frac{1}{\cos x}\right)'=\left(\frac{\sin x+1}{\cos x}\right)'$

$=\frac{(\sin x+1)'\cos x-(\sin x+1)(\cos x)'}{\cos^2 x}=\frac{\cos^2 x+(1+\sin x)\sin x}{\cos^2 x}$

$=\frac{\sin x+\sin^2 x+\cos^2 x}{\cos^2 x}=\frac{1+\sin x}{\cos^2 x}\cdot\frac{1-\sin x}{1-\sin x}$

$=\frac{1-\sin^2 x}{\cos^2 x(1-\sin x)}=\frac{\cos^2 x}{\cos^2 x(1-\sin x)}=\frac{1}{1-\sin x}$;

7) $f'(x)=\left(\frac{x+\sin x}{x-\sin x}\right)'=\frac{(x+\sin x)'(x-\sin x)-(x+\sin x)(x-\sin x)'}{(x-\sin x)^2}=$

$\frac{(1+\cos x)(x-\sin x)-(x+\sin x)(1-\cos x)}{(x-\sin x)^2}$

$=\frac{x-\sin x+x\cos x-\sin x\cos x-x+x\cos x-\sin x+\sin x\cos x}{(x-\sin x)^2}=\frac{2(x\cos x-\sin x)}{(x-\sin x)^2}$;

8) $f'(x)=\left(\frac{x+\operatorname{tg}x}{x-\operatorname{tg}x}\right)'=\left(\frac{x+\frac{\sin x}{\cos x}}{x-\frac{\sin x}{\cos x}}\right)'=\left(\frac{x\cos x+\sin x}{x\cos x-\sin x}\right)'$

$=\frac{(x\cos x+\sin x)'(x\cos x-\sin x)-(x\cos x+\sin x)(x\cos x-\sin x)'}{(x\cos x-\sin x)^2}$

$=\frac{[(x\cos x)'+(\sin x)'](x\cos x-\sin x)-(x\cos x+\sin x)[(x\cos x)'-(\sin x)']}{(x\cos x-\sin x)^2}$

$=\frac{[x'\cos x+x(\cos x)'+\cos x](x\cos x-\sin x)-(x\cos x+\sin x)[x'\cos x+x(\cos x)'-\cos x]}{(x\cos x-\sin x)^2}$

$=\frac{(\cos x-x\sin x+\cos x)(x\cos x-\sin x)-(x\cos x+\sin x)(\cos x-x\sin x-\cos x)}{(x\cos x-\sin x)^2}$

$=\frac{(2\cos x-x\sin x)(x\cos x-\sin x)-(x\cos x+\sin x)(-x\sin x)}{(x\cos x-\sin x)^2}$

$=\frac{2x\cos^2 x-2\sin x\cos x-x^2\sin x\cos x+x\sin^2 x+x^2\cos x\sin x+x\sin^2 x}{(x\cos x-\sin x)^2}$

$=\frac{2x\cos^2 x-2\sin x\cos x+2x\sin^2 x}{(x\cos x-\sin x)^2}=\frac{2x(\cos^2 x+\sin^2 x)-\sin 2x}{(x\cos x-\sin x)^2}$

$=\frac{2x-\sin 2x}{(x\cos x-\sin x)^2}$;

9) $f'(x)=\left(\frac{\sin x-\cos x}{\sin x+\cos x}\right)$

$=\frac{(\sin x-\cos x)'(\sin x+\cos x)-(\sin x-\cos x)(\sin x+\cos x)'}{(\sin x+\cos x)^2}$

$=\frac{[(\sin x)'-(\cos x)](\sin x+\cos x)-(\sin x-\cos x)[(\sin x)'+(\cos x)']}{(\sin x+\cos x)^2}$

$=\frac{(\cos x+\sin x)(\sin x+\cos x)-(\sin x-\cos x)(\cos x-\sin x)}{(\sin x+\cos x)^2}$

$=\frac{(\cos x+\sin x)^2+(\sin x-\cos x)^2}{(\sin x+\cos x)^2}$

$= \dfrac{\cos^2 x + 2\sin x\cos x + \sin^2 x + \sin^2 x - 2\sin x\cos x + \cos^2 x}{(\sin^2 x + 2\sin x\cos x + \cos^2 x} = \dfrac{2}{1 + 2\sin x\cos x} = \dfrac{2}{1+\sin 2x}$;

10) $f'(x) = \left(\dfrac{\operatorname{tg} x - \operatorname{ctg} x}{\operatorname{tg} x + \operatorname{ctg} x}\right)' = \left(\dfrac{\frac{\sin x}{\cos x} - \frac{\cos x}{\sin x}}{\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}}\right) = \left(\dfrac{\frac{\sin^2 x - \cos^2 x}{\sin x\cos x}}{\frac{\sin^2 x + \cos^2 x}{\sin x\cos x}}\right) =$ $(\sin^2 x - \cos^2 x)' = [(\sin x - \cos x)(\sin x + \cos x)]' = (\sin x - \cos x)'(\sin x + \cos x) + (\sin x - \cos x)(\sin x + \cos x)' = [(\sin x)' - (\cos x)'](\sin x + \cos x) + (\sin x - \cos x)[(\sin x)' + (\cos x)'] = (\cos x + \sin x)(\sin x + \cos x) + (\sin x - \cos x)(\cos x - \sin x) = (\sin x + \cos x)^2 - (\sin x - \cos x)^2 = \sin^2 x + 2\sin x\cos x + \cos^2 x - \sin^2 x + 2\sin x\cos x - \cos^2 x = 4\sin x\cos x = 2\sin 2x$.