

Zadatak 17. Odredi ekstreme sljedećih funkcija:

1) $f(x) = x + \cos x + \sin x$;

2) $f(x) = \frac{1}{2} \sin 2x + \cos x$;

3) $f(x) = \frac{1 - \cos x}{\sin x + \cos x}$;

4) $f(x) = \frac{\sin x + \cos x}{\cos 2x}$.

Rješenje. 1) $f'(x) = 1 - \sin x + \cos x$, $f''(x) = -\cos x - \sin x$. Stacionarne točke računamo iz $1 - \sin x + \cos x = 0 \implies \sin x - \cos x = 1/2 \implies 1 - \sin 2x = 1 \implies \sin 2x = 0 \implies 2x = k\pi \implies x = k\frac{\pi}{2}$, $k \in \mathbf{Z}$.

$f'(0) = 2$, $f'(\frac{\pi}{2}) = 0$, $f'(\pi) = 0$, $f'(\frac{3\pi}{2}) = 2 \implies x_1 = \frac{\pi}{2} + 2k\pi$ i $x_2 = (2k+1)\pi$, $k \in \mathbf{Z}$.

$f''(\frac{\pi}{2}) = -1 < 0$ i $f''(\pi) = 1 > 0$, slijedi: funkcija postiže maksimum u točkama $x = \frac{\pi}{2} + 2k\pi$, a minimum u $x = (2k+1)\pi$.

2) $f'(x) = \cos 2x - \sin x = \cos^2 x - \sin^2 x - \sin x = 1 - 2\sin^2 x - \sin x = 1 + \sin x - 2\sin x - 2\sin^2 x = (1 - 2\sin x)(1 + \sin x)$, $f''(x) = -2\sin 2x - \cos x$. Stacionarne točke računamo iz $(1 - 2\sin x)(1 + \sin x) = 0 \implies \sin x = \frac{1}{2}$ ili $\sin x = -1$. Odavde slijedi $x = \frac{\pi}{6} + 2k\pi$ i $x = \frac{3\pi}{2} + 2k\pi$.

$f''(\frac{\pi}{6}) < 0$ i $f''(\frac{5\pi}{6}) > 0$. Funkcija ima maksimum u točkama $\frac{\pi}{6} + 2k\pi$, a minimum u točkama $\frac{5\pi}{6} + 2k\pi$, $k \in \mathbf{Z}$.

$$\begin{aligned} 3) f'(x) &= \frac{\sin x(\sin x + \cos x) - (1 - \cos x)(\cos x - \sin x)}{(\sin x + \cos x)^2} \\ &= \frac{\sin^2 x + \sin x \cos x - \cos x + \cos^2 x + \sin x - \sin x \cos x}{1 + \sin 2x} \\ &= \frac{1 + \sin x - \cos x}{1 + \sin 2x}, \end{aligned}$$

$f''(x) = \frac{(\cos x + \sin x)(1 + \sin 2x) + (1 + \sin x - \cos x)(2\cos 2x)}{(1 + \sin 2x)^2}$. Stacionarne točke računamo iz $1 + \sin x - \cos x = 0 \implies 2\sin^2 \frac{x}{2} + 2\sin \frac{x}{2} \cos \frac{x}{2} = 0 \implies 2\sin \frac{x}{2} \left(\sin \frac{x}{2} + \cos \frac{x}{2} \right) = 0$. Odavde slijedi: $\sin \frac{x}{2} = 0 \implies \frac{x}{2} = k\pi \implies x = 2k\pi$, $k \in \mathbf{Z}$ i $\sin \frac{x}{2} + \cos \frac{x}{2} = 0 \implies \sin \frac{x}{2} = -\cos \frac{x}{2} \implies \frac{x}{2} = \frac{3\pi}{4} + k\pi \implies x = \frac{3\pi}{2} + 2k\pi$, $k \in \mathbf{Z}$.

$f''(2k\pi) > 0$ i $f''(\frac{3\pi}{2} + 2k\pi) < 0$ pa slijedi da funkcija postiže minimum u točkama $2k\pi$, $k \in \mathbf{Z}$, a maksimum u točkama $\frac{3\pi}{2} + 2k\pi$, $k \in \mathbf{Z}$.

$$4) f(x) = \frac{\sin x + \cos x}{\cos 2x} = \frac{\sin x + \cos x}{\cos^2 x - \sin^2 x} = \frac{1}{\cos x - \sin x}.$$
$$f'(x) = \frac{\sin x + \cos x}{(\cos x - \sin x)^2} = \frac{\sin x + \cos x}{1 - \sin 2x},$$

$$f''(x) = \frac{(\cos x - \sin x)^3 + (\sin x + \cos x)2 \cos 2x}{(1 - \sin 2x)^2}.$$

stacionarne točke računamo iz $\sin x + \cos x = 0 \implies x = \frac{3\pi}{4} + k\pi, k \in \mathbf{Z}$.

$f''\left(\frac{3\pi}{4} + 2k\pi\right) < 0$ i $f''\left(\frac{7\pi}{4} + 2k\pi\right) > 0$ pa slijedi da funkcija postiže svoj maksimum u točkama $\frac{3\pi}{4} + 2k\pi, k \in \mathbf{Z}$, a minimum u točkama $\frac{7\pi}{4} + 2k\pi, k \in \mathbf{Z}$.