Zadatak 20. Nađi najmanju i najveću vrijednost funkcije f na danom intervalu:

1) $f(x) = \sin 2x - x$, $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$;

2) $f(x) = \sin x \cdot \cos^2 \frac{x}{2}$, $x \in [0, \pi]$;

3) $f(x) = \operatorname{tg} x + \operatorname{ctg} x$, $x \in [\frac{\pi}{6}, \frac{\pi}{3}]$;

4) $f(x) = \dfrac{\sin 2x}{\sin(\frac{\pi}{4} + x)}$, $x \in [\pi, \frac{3\pi}{2}]$;

5) $f(x) = \frac{1}{2}\cos 2x + \sin x$, $x \in [0, \frac{\pi}{2}]$;

6) $f(x) = 2\sin 2x + \cos 4x$, $x \in [0, \frac{\pi}{3}]$.

Rješenje. **1)** $f'(x) = 2\cos 2x - 1$, $2\cos 2x - 1 = 0 \implies \cos 2x = \frac{1}{2} \implies 2x = \pm\frac{\pi}{3} \implies x = \pm\frac{\pi}{6}$.

$f''(x) = -4\sin 4x$; $f''\left(\frac{\pi}{6}\right) = -2\sqrt{3}$, $f''\left(-\frac{\pi}{6}\right) = 2\sqrt{3}$.

$f\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} - \frac{\pi}{6}$, $f\left(-\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2} + \frac{\pi}{6}$, $f\left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$, $f\left(\frac{\pi}{2}\right) = -\frac{\pi}{2}$.

$M\left(\frac{\pi}{6}, \frac{3\sqrt{3}-\pi}{6}\right)$, $m\left(-\frac{\pi}{6}, \frac{-3\sqrt{3}+\pi}{6}\right)$.

2) $f'(x) = \cos x \cos^2 \frac{x}{2} - 2\sin x \cos \frac{x}{2} \sin \frac{x}{2} = \cos x \cos^2 \frac{x}{2} - \sin^2 x = \cos x \frac{1}{2}(1 - \cos x) - \sin^2 x = \frac{1}{2}\cos x - \frac{1}{2}\cos^2 x - 1 + \cos^2 x = \frac{1}{2}\cos x + \frac{1}{2}\cos^2 x - 1 = \frac{1}{2}(\cos^2 x + \cos x - 1)$, $\cos^2 x + \cos x - 1 = 0 \implies (\cos x)_{1,2} = \dfrac{-1 \pm \sqrt{1+4}}{2} = -\dfrac{1 \pm \sqrt{5}}{2} \implies \cos x = \dfrac{-1+\sqrt{5}}{2} \implies x_1 = 51°49'38.25''$.

$f''(x) = \frac{1}{2}(-2\cos x \sin x - \sin x) = -\frac{1}{2}(\sin 2x + \sin x)$, $f''(51°49'38.25'') < 0$, $x = 51°49'38.25''$ je točka lokalnog maksimuma.

$f(51°49'38.25'') = 0.70711$, $f(0) = 0$, $f(\pi) = 0$.

$m(0,0)$ i $m(\pi, 0)$, $M(51°49'38'', 0.70711)$.

3) $f(x) = \dfrac{\sin x}{\cos x} + \dfrac{\cos x}{\sin x} = \dfrac{\sin^2 x + \cos^2 x}{\sin x \cos x} = \dfrac{2}{\sin 2x}$.

$f'(x) = -\dfrac{4\cos 2x}{\sin^2 2x}$, $-\dfrac{4\cos 2x}{\sin^2 2x} = 0 \implies \cos 2x = 0 \implies 2x = \dfrac{\pi}{2} \implies x = \dfrac{\pi}{4}$.

$f''(x) = \dfrac{\sin x}{\cos^3 x} - \dfrac{\cos x}{\sin^3 x}$, $f''\left(\frac{\pi}{4}\right) = 0$.

$f\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{3} + \sqrt{3}$, $f\left(\frac{\pi}{3}\right) = \sqrt{3} + \frac{\sqrt{3}}{3} = \frac{4\sqrt{3}}{3}$.

$\sin 2x \leqslant 1 \implies \dfrac{1}{\sin 2x} \geqslant 1 \implies \dfrac{2}{\sin 2x} \geqslant 2$

$\dfrac{2}{\sin 2x} = 2 \implies \sin 2x = 1 \implies 2x = \dfrac{\pi}{2} \implies x = \dfrac{\pi}{4}.$

$M\left(\dfrac{\pi}{6}, \dfrac{4\sqrt{3}}{3}\right)$, $M\left(\dfrac{\pi}{3}, \dfrac{4\sqrt{3}}{3}\right)$, i $m\left(\dfrac{\pi}{4}, 2\right)$

4) $f(x) = \dfrac{\sin 2x}{\sin\left(\dfrac{\pi}{4} + x\right)} = \dfrac{\sin 2x}{\sin\dfrac{\pi}{4}\cos x + \sin x \cos\dfrac{\pi}{4}} = \dfrac{sin2x}{\dfrac{\sqrt{2}}{2}(\sin x + \cos x)} =$

$\sqrt{2}\dfrac{\sin 2x}{\sin x + \cos x}.$

$f'(x) = \sqrt{2}\dfrac{2\cos 2x(\sin x + \cos x) - \sin 2x(\cos x - \sin x)}{1 + \sin 2x}$

$= \sqrt{2}\dfrac{(\cos x - \sin x)[2(1 + \sin 2x) - \sin 2x]}{1 + \sin 2x}$

$= \sqrt{2}\dfrac{(\cos x - \sin x)(2 + \sin 2x)}{1 + \sin 2x}.$

$2 + \sin 2x > 0,\ \forall x,\ \cos x - \sin x = 0 \implies \sin x = \cos x \implies x = \dfrac{5\pi}{4}.$

$f\left(\dfrac{5\pi}{4}\right) = -1,\ f(\pi) = 0,\ f\left(\dfrac{3\pi}{2}\right) = 0,\ f(x) \leqslant 0,\ \forall x \in \left[\pi, \dfrac{3\pi}{2}\right].$

$M(\pi, 0),\ M\left(\dfrac{3\pi}{2}, 0\right),\ m\left(\dfrac{5\pi}{4}, -1\right).$

5) $f'(x) = \dfrac{1}{2}(-2\sin 2x) + \cos x = -\sin 2x + \cos x = -2\sin x\cos x + \cos x =$

$\cos x(1 - 2\sin x),\ f''(x) = -2\cos 2x - \sin x.$

$\cos x(1 - 2\sin x) = 0 \implies x_1 = \dfrac{\pi}{2},\ x_2 = \dfrac{\pi}{6}.$

$f\left(\dfrac{\pi}{2}\right) = \dfrac{1}{2},\ f\left(\dfrac{\pi}{6}\right) = \dfrac{3}{4}.$

$f''\left(\dfrac{\pi}{2}\right) = 1,\ f''\left(\dfrac{\pi}{6}\right) = -\dfrac{3}{2},\ f(0) = \dfrac{1}{2}.$

$M\left(\dfrac{\pi}{6}, \dfrac{3}{4}\right),\ m\left(0, \dfrac{1}{2}\right),\ m\left(\dfrac{\pi}{2}, \dfrac{1}{2}\right).$

6) $f'(x) = 4\cos 2x - 4\sin 4x = 4\cos 2x - 8\sin 2x\cos 2x = 4\cos 2x(1 - 2\sin 2x).$

$\cos 2x = 0 \implies x = \dfrac{\pi}{4}$ i $\sin 2x = \dfrac{1}{2} \implies x = \dfrac{\pi}{12}.$

$f\left(\dfrac{\pi}{4}\right) = 1,\ f\left(\dfrac{\pi}{12}\right) = \dfrac{3}{2},\ f(0) = 1,\ f\left(\dfrac{\pi}{3}\right) = \sqrt{3} - \dfrac{1}{2} \approx 1.23.$

$m(0, 1),\ m\left(\dfrac{\pi}{4}, 1\right),\ M\left(\dfrac{\pi}{12}, \dfrac{3}{2}\right).$