Rješenja zadataka 4.7

Zadatak 1. Grafički prikaži sljedeće funkcije:

1) $f(x) = x^3 - 3x + 2$;

2) $f(x) = x^3 - 4x^2 - 3x + 12$;

3) $f(x) = x^3 + 3x^2 + 2x$;

4) $f(x) = (x^2 + x)(x - 2)$;

5) $f(x) = x^3 - 3x^2 + 4$;

6) $f(x) = x^3 - \frac{3}{2}x^2$.

Rješenje. **1)** $f(x) = x^3 - 3x + 2$,

$D_f = \mathbf{R}$

$\lim\limits_{x\to\pm\infty}(x^3 - 3x + 2) = \pm\infty \implies$ nema asimptota

$\lim\limits_{x\to\pm\infty}\dfrac{x^3 - 3x + 2}{x} = +\infty \implies$ nema asimptota

$x^3 - 3x + 2 = 0$

	1	0	−3	2
1	1	1	−2	0

$(x-1)(x^2 + x - 2) = 0$

$(x-1)^2(x+2) = 0 \implies x_1 = -2, \quad x_{2,3} = 1$ nultočke

x	$\langle -\infty, -2\rangle$	$\langle -2, 1\rangle$	$\langle 1, +\infty\rangle$
$f(x)$	−	+	+

$f'(x) = 3x^2 - 3 = 3(x^2 - 1)$

$f'(x) = 0 \implies x_1 = 1,\ x_2 = -1,\ f(1) = 0,\ f(-1) = 4$

x	$\langle -\infty, -1\rangle$	$\langle -1, 1\rangle$	$\langle 1, +\infty\rangle$
$f'(x)$	+	−	+

$f''(x) = 6x$

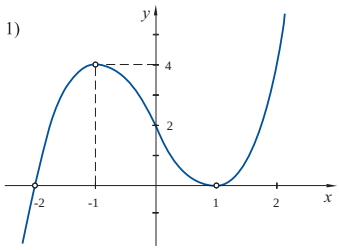
$f''(1) > 0,\ f''(-1) < 0 \implies m(1, 0),\ M(-1, 4)$

$f''(x) = 0 \implies x = 0$

x	$\langle -\infty, 0\rangle$	$\langle 0, +\infty\rangle$
$f''(x)$	−	+

x	$\langle -\infty, -2\rangle$	-2	$\langle -2, -1\rangle$	-1	$\langle -1, 0\rangle$	0	$\langle 0, 1\rangle$	1	$\langle 1, +\infty\rangle$
$f(x)$	−	0	+	4	+	2	+	0	+
$f'(x)$	+	+	+	0	−	−	−	0	+
$f''(x)$	−	−	−	−	−	0	+	+	+
	negat. uzlazna konkav.	nul-točka	pozit. uzlaz. konk.	max	pozit. silazna konk.	infleksija	pozit. silazna konveks.	min	pozit. uzlaz konveks.

1)



2) $f(x) = x^3 - 4x^2 - 3x + 12$,

$D_f = \mathbf{R}$

$\lim\limits_{x\to\pm\infty} f(x) = \pm\infty \implies$ nema asimptota

$\lim\limits_{x\to\pm\infty} \dfrac{f(x)}{x} = \infty \implies$ nema asimptota

$x^3 - 4x^2 - 3x + 12 = x^2(x-4) - 3(x-4) = (x-\sqrt{3})(x+\sqrt{3})(x-4)$

$\implies x_1 = -\sqrt{3}, \quad x_2 = \sqrt{3}, \quad x_3 = 4$ nultočke

x	$\langle -\infty, -\sqrt{3}\rangle$	$\langle -\sqrt{3}, \sqrt{3}\rangle$	$\langle \sqrt{3}, 4\rangle$	$\langle 4, +\infty\rangle$
$f(x)$	$-$	$+$	$-$	$+$

$f'(x) = 3x^2 - 8x - 3 = 3x^2 - 9x + x - 3 = (3x+1)(x-3)$

$f'(x) = 0 \implies x_1 = -\dfrac{1}{3}, \quad x_2 = 3, \quad f\left(-\dfrac{1}{3}\right) = \dfrac{338}{27} \approx 12.5, \quad f(3) = -6$

x	$\left\langle -\infty, -\frac{1}{3}\right\rangle$	$\left\langle -\frac{1}{3}, 3\right\rangle$	$\langle 3, +\infty\rangle$
$f'(x)$	$+$	$-$	$+$

$f''(x) = 6x - 8$

$f''\left(-\dfrac{1}{3}\right) < 0, \quad f''(3) > 0$

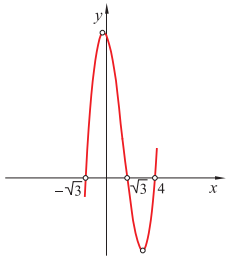
$f''(x) = 0 \implies x = \dfrac{4}{3}, \quad f\left(\dfrac{4}{3}\right) = \dfrac{88}{27} \approx 3.26$

x	$\left\langle -\infty, \frac{4}{3}\right\rangle$	$\left\langle \frac{4}{3}, +\infty\right\rangle$
$f''(x)$	$-$	$+$

x	$\langle -\infty, -\sqrt{3}\rangle$	$-\sqrt{3}$	$\left\langle -\sqrt{3}, -\frac{1}{3}\right\rangle$	$-\frac{1}{3}$	$\left\langle -\frac{1}{3}, \frac{4}{3}\right\rangle$	$\frac{4}{3}$
$f(x)$	$-$	0	$+$	$\frac{338}{27}$	$+$	$\frac{88}{27}$
$f'(x)$	$+$	$+$	$+$	0	$-$	$-$
$f''(x)$	$-$	$-$	$-$	$-$	$-$	0
	neg. uzlazna konk.	nul-točka	pozit. uzl. konk.	max	pozit. silaz. konk.	infl.

x	$\left\langle \frac{4}{3}, \sqrt{3} \right\rangle$	$\sqrt{3}$	$\langle \sqrt{3}, 3 \rangle$	3	$\langle 3, 4 \rangle$	4	$\langle 4, +\infty \rangle$
$f(x)$	+	0	−	−6	−	0	+
$f'(x)$	−	−	−	0	+	+	+
$f''(x)$	+	+	+	+	+	+	+
	pozit. silaz, konvek.	nul-točka	neg. silaz. konv.	min	neg. uzl. konv.	nul-točla	pozit. uzlaz. konk.

2)



3) $f(x) = x^3 + 3x^2 + 2x$,

$D_f = \mathbf{R}$

$\lim\limits_{x \to \pm\infty} f(x) = \pm\infty \implies$ nema asimptota

$\lim\limits_{x \to \pm\infty} \frac{f(x)}{x} = \pm\infty \implies$ nema asimptota

$x^3 + 3x^2 + 2x = x(x+1)(x+2) \implies x_1 = -2, \quad x_2 = -1, \quad x_3 = 0 \implies$ nultočke

x	$\langle -\infty, -2 \rangle$	$\langle -2, -1 \rangle$	$\langle -1, 0 \rangle$	$\langle 0, +\infty \rangle$
$f(x)$	−	+	−	+

$f'(x) = 3x^2 + 6x + 2$

$$x_{1,2} = \frac{-6 \pm \sqrt{36 - 4 \cdot 3 \cdot 2}}{6} = \frac{-6 \pm 2\sqrt{3}}{3} = \frac{-3 \pm \sqrt{3}}{3} = -1 \pm \frac{\sqrt{3}}{3}$$

$$f\left(-1 - \frac{\sqrt{3}}{3}\right) \approx 0.4, \quad f\left(-1 + \frac{\sqrt{3}}{3}\right) \approx -0.4$$

x	$\left\langle -\infty, -1 - \frac{\sqrt{3}}{3} \right\rangle$	$\left\langle -1 - \frac{\sqrt{3}}{3}, -1 + \frac{\sqrt{3}}{3} \right\rangle$	$\left\langle -1 + \frac{\sqrt{3}}{3}, +\infty \right\rangle$
$f'(x)$	+	−	+

$f''(x) = 6x + 6 = 6(x+1)$

$f''(x_1) < 0, \quad f''(x_2) > 0$

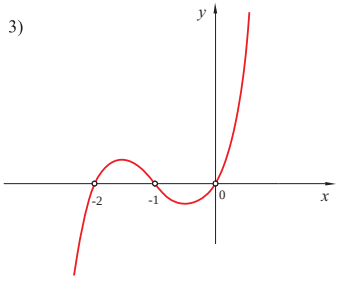
$f''(x) = 0 \implies x = -1, \quad f(-1) = 0$

x	$\langle -\infty, -1 \rangle$	$\langle -1, +\infty \rangle$
$f(x)$	−	+

x	$\langle -\infty, -2\rangle$	-2	$\langle -2, -1-\frac{\sqrt{3}}{3}\rangle$	$-1-\frac{\sqrt{3}}{3}$	$\langle -1-\frac{\sqrt{3}}{3}, -1\rangle$	-1
$f(x)$	$-$	0	$+$	0.4	$+$	0
$f'(x)$	$+$	$+$	$+$	0	$-$	$-$
$f''(x)$	$-$	$-$	$-$	$-$	$-$	0
	neg. uzl. konk.	nul-točka	pozit. uzl. konk.	max	pozit. sil. konk.	nul-točka inflek.

x	$\langle -1, -1+\frac{\sqrt{3}}{3}\rangle$	$-1+\frac{\sqrt{3}}{3}$	$\langle -1+\frac{\sqrt{3}}{3}, 0\rangle$	0	$\langle 0, +\infty\rangle$
$f(x)$	$-$	-0.4	$-$	0	$+$
$f'(x)$	$-$	0	$+$	$+$	$+$
$f''(x)$	$+$	$+$	$+$	$+$	$+$
	neg. sil. konv.	min	neg. uzl. konv.	nul-točka	pozit. uzlaz. konvek.

3)



4) $f(x) = (x^2+x)(x-2) = x(x+1)(x-2) = x^3 - x^2 - 2x,$

$D_f = \mathbf{R}$

$\lim\limits_{x\to\pm\infty} f(x) = \pm\infty \implies$ nema asimptota

$\lim\limits_{x\to\pm\infty} \dfrac{f(x)}{x} = \infty \implies$ nema asimptota

$x_1 = -1, \quad x_2 = 0, \quad x_3 = 2$ nultočke

x	$\langle -\infty, -1\rangle$	$\langle -1, 0\rangle$	$\langle 0, 2\rangle$	$\langle 2, +\infty\rangle$
$f(x)$	$-$	$+$	$-$	$+$

$f'(x) = 3x^2 - 2x - 2$

$3x^2 - 2x - 2 = 0$

$x_{1,2} = \dfrac{2 \pm \sqrt{4+24}}{6} = \dfrac{2 \pm 2\sqrt{7}}{6} = \dfrac{1 \pm \sqrt{7}}{3}, \quad x_1 \approx -0.55, \quad x_2 \approx 1.22$

$f(x_1) \approx 0.63, \quad f(x_2) \approx -2.1$

x	$\left\langle -\infty, \frac{1-\sqrt{7}}{3}\right\rangle$	$\left\langle \frac{1-\sqrt{7}}{3}, \frac{1+\sqrt{7}}{3}\right\rangle$	$\left\langle \frac{1+\sqrt{7}}{3}, +\infty\right\rangle$
$f'(x)$	$+$	$-$	$+$

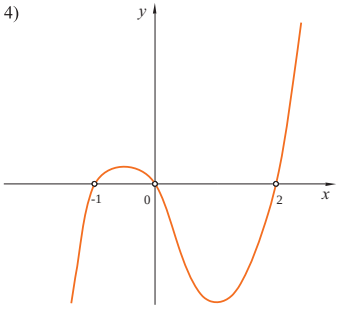
$$f''(x) = 6x - 2 = 2(3x - 1)$$
$$f''(-0.55) < 0, \quad f''(1.22) > 0$$
$$f''(x) = 0 \implies x = \frac{1}{3}, \quad f\left(\frac{1}{3}\right) = -\frac{20}{27}$$

x	$\left\langle -\infty, \frac{1}{3} \right\rangle$	$\left\langle \frac{1}{3}, +\infty \right\rangle$
$f(x)$	$-$	$+$

x	$\langle -\infty, -1\rangle$	-1	$\langle -1, \frac{1-\sqrt{7}}{3}\rangle$	$\frac{1-\sqrt{7}}{3}$	$\langle \frac{1-\sqrt{7}}{3}, 0\rangle$	0	$\langle 0, \frac{1}{3}\rangle$
$f(x)$	$-$	0	$+$	0.63	$+$	0	$-$
$f'(x)$	$+$	$+$	$+$	0	$-$	$-$	$-$
$f''(x)$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
	neg. uzl. konk.	nul-točka	pozit. uzl. konk.	max	pozit. sil. konk.	nul-točka	negat. sil. konk.

x	$\frac{1}{3}$	$\langle \frac{1}{3}, \frac{1+\sqrt{7}}{3}\rangle$	$\frac{1+\sqrt{7}}{3}$	$\langle \frac{1+\sqrt{7}}{3}, 2\rangle$	2	$\langle 2, +\infty\rangle$
$f(x)$	$-\frac{20}{27}$	$-$	-2.1	$-$	0	$+$
$f'(x)$	$-$	$-$	0	$+$	$+$	$+$
$f''(x)$	0	$+$	$+$	$+$	$+$	$+$
	inflek.	neg. sil. konk.	min	neg. uzl. konk.	nul-točka	pozit. uzl. konv

4)



5) $f(x) = x^3 - 3x^2 + 4$

$$D_f = \mathbf{R} \implies \text{nema vertikalnih asimptota}$$
$$\lim_{x \to \pm\infty} f(x) = \pm\infty \implies \text{nema horizontalnih asimptota}$$
$$\lim_{x \to \pm\infty} \frac{f(x)}{x} = +\infty \implies \text{nema kosih asimptota}$$
$$x^3 - 3x^2 + 4 = 0$$

	1	-3	0	4
2	1	-1	-2	0

$$(x-2)(x^2 - x - 2) = 0 \iff (x-2)^2(x+1) = 0 \implies x_1 = -1, \quad x_{2,3} = 2$$

x	$\langle -\infty, -1\rangle$	$\langle -1, 2\rangle$	$\langle 2, +\infty\rangle$
$f(x)$	$-$	$+$	$+$

$$f'(x) = 3x^2 - 6x = 3x(x-2)$$
$$f'(x) = 0 \implies x_1 = 0, \quad x_2 = 2, \quad f(0) = 4, \quad f(2) = 0$$

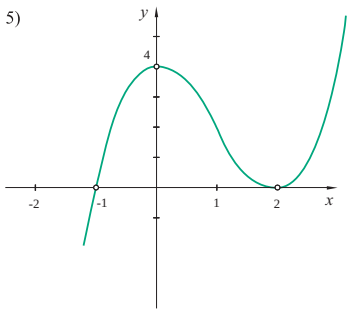
x	$\langle -\infty, 0\rangle$	$\langle 0, 2\rangle$	$\langle 2, +\infty\rangle$
$f'(x)$	$+$	$-$	$+$

$$f''(x) = 6x - 6 = 6(x-1)$$
$$f''(0) = -6 < 0, \quad f''(2) = 6 > 0 \implies M(0,4), \quad m(2,0)$$
$$f''(x) = 0 \implies 6x - 6 = 0 \implies x = 1, \quad f(1) = 2 \implies (1,2) \quad \text{infleksija}$$

x	$\langle -\infty, 1\rangle$	$\langle 1, +\infty\rangle$
$f''(x)$	$-$	$+$

x	$\langle -\infty, -1\rangle$	-1	$\langle -1, 0\rangle$	0	$\langle 0, 1\rangle$	1	$\langle 1, 2\rangle$	2	$\langle 2, +\infty\rangle$
$f(x)$	$-$	0	$+$	4	$+$	2	$+$	0	$+$
$f'(x)$	$+$	$+$	$+$	0	$-$	$-$	$-$	0	$+$
$f''(x)$	$-$	$-$	$-$	$-$	$-$	0	$+$	$+$	$+$
	negat. uzlaz. konk.	nul-točka	pozit. uzlaz. konk.	max	pozit. silaz. konk.	inflek.	pozit. silaz. konv.	nul-točka i min	pozit. uzlaz. konv.

5)



6) $f(x) = x^3 - \frac{3}{2}x^2$,

$$D_f = \mathbf{R} \implies \text{nema vertikalnih asimptota}$$
$$\lim_{x\to\pm\infty} f(x) = \pm\infty \implies \text{nema horizontalnih asimptota}$$
$$\lim_{x\to\pm\infty} \frac{f(x)}{x} = +\infty \implies \text{nema kosih asimptota}$$
$$x^3 - \frac{3}{2}x^2 = 0 \implies x^2\left(x - \frac{3}{2}\right) = 0 \implies x_{1,2} = 0, \quad x_3 = \frac{3}{2}$$

x	$\langle -\infty, 0\rangle$	$\left\langle 0, \frac{3}{2}\right\rangle$	$\left\langle \frac{3}{2}, +\infty\right\rangle$
$f(x)$	$-$	$-$	$+$

$$f'(x) = 3x^2 - 3x = 3x(x-1)$$

$$f'(x) = 0 \implies x_1 = 0, \quad x_2 = 1, \quad f(0) = 0, \quad f(1) = -\frac{1}{2}$$

x	$\langle -\infty, 0\rangle$	$\langle 0, 1\rangle$	$\langle 1, +\infty\rangle$
$f'(x)$	$+$	$-$	$+$

$$f''(x) = 6x - 3 = 3(2x-1)$$

$$f''(0) = -3 \implies M(0,0), \quad f''(1) = 3 \implies m\left(1, -\frac{1}{2}\right)$$

$$f''(x) = 0 \implies x = \frac{1}{2}, \quad f\left(\frac{1}{2}\right) = -\frac{1}{4}$$

x	$\left\langle -\infty, \frac{1}{2}\right\rangle$	$\left\langle \frac{1}{2}, +\infty\right\rangle$
$f'(x)$	$-$	$+$

x	$\langle -\infty, 0\rangle$	0	$\left\langle 0, \frac{1}{2}\right\rangle$	$\frac{1}{2}$	$\left\langle \frac{1}{2}, 1\right\rangle$	1	$\left\langle 1, \frac{3}{2}\right\rangle$	$\frac{3}{2}$	$\left\langle \frac{3}{2}, +\infty\right\rangle$
$f(x)$	$-$	0	$-$	$-\frac{1}{4}$	$-$	$-\frac{1}{2}$	$-$	0	$+$
$f'(x)$	$+$	0	$-$	$-$	$-$	0	$+$	$+$	$+$
$f''(x)$	$-$	-3	$-$	0	$+$	3	$+$	$+$	$+$
	neg. uzlaz. konk.	nultočka i max	neg. silaz. konk.	infleks.	neg. silaz. konv.	min	neg. uzlaz. konv.	nul-točka	pozit. uzlaz. konv.

6)