Zadatak 2. Grafički prikaži sljedeće funkcije:

1) $f(x) = (x-1)^2(x+2)^2$;

2) $f(x) = (x-1)^2(x+1)^2$;

3) $f(x) = 3x^4 - 4x^3 + 1$;

4) $f(x) = x(x+1)(x-1)(x-2)$;

5) $f(x) = x^4 + x$;

6) $f(x) = x^4 - 4x^2$.

Rješenje. **1)** $f(x) = (x-1)^2(x+2)^2$,

$$D_f = \mathbf{R} \implies \text{nema vertikalnih asimptota}$$

$$\lim_{x\to\pm\infty} f(x) = +\infty \implies \text{nema horizontalnih asimptota}$$

$$\lim_{x\to\pm\infty} \frac{f(x)}{x} = \pm\infty \implies \text{nema kosih asimptota}$$

$$f(x) = 0 \implies x_{1,2} = 1, \quad x_{3,4} = -2$$

$$f(x) \geqslant 0, \forall x \in \mathbf{R}$$

$$\begin{aligned} f'(x) &= 2(x-1)(x+2)^2 + 2(x-1)^2(x+2) = 2(x-1)(x+2)(x+2+x-1) \\ &= 2(x-1)(x+2)(2x+1) \end{aligned}$$

$$f'(x) = 0 \implies x_1 = 1, \quad x_2 = -2, \quad x_3 = -\frac{1}{2}, \quad f(x_3) = \frac{81}{16}$$

x	$\langle -\infty, -2\rangle$	$\left\langle -2, -\frac{1}{2}\right\rangle$	$\left\langle -\frac{1}{2}, 1\right\rangle$	$\langle 1, +\infty\rangle$
$f'(x)$	$-$	$+$	$-$	$+$

$$\begin{aligned} f''(x) &= 2\{[(x-1)+(x+2)](2x+1) + (x-1)(x+2)\cdot 2\} \\ &= 2\{(2x+1)^2 + 2(x^2+x-2)\} \\ &= 2\{4x^2+4x+1+2x^2+2x-4\} = 2(6x^2+6x-3) \end{aligned}$$

$$f''(x) = 6(2x^2+2x-1)$$

$$f''(1) > 0, \quad f''(-2) > 0, \quad f''\left(-\frac{1}{2}\right) < 0$$

$$f''(x) = 0 \implies 2x^2+2x-1 = 0$$

$$x_{1,2} = \frac{-2\pm\sqrt{4+8}}{4} = \frac{-2\pm 2\sqrt{3}}{4} = \frac{-1\pm\sqrt{3}}{2}$$

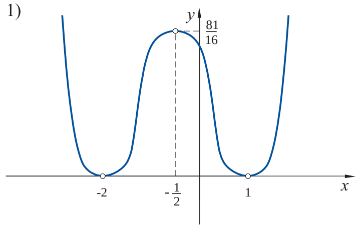
$$f\left(\frac{-1-\sqrt{3}}{2}\right) = \left(\frac{-3-\sqrt{3}}{2}\right)^2\left(\frac{3-\sqrt{3}}{2}\right)^2 = \left(\frac{3}{4}-\frac{9}{4}\right)^2 = \left(-\frac{3}{2}\right)^2 = \frac{9}{4}$$

$$f\left(\frac{-1+\sqrt{3}}{2}\right) = \left(\frac{-3+\sqrt{3}}{2}\right)^2\left(\frac{3+\sqrt{3}}{2}\right)^2 = \left(\frac{3}{4}-\frac{9}{4}\right)^2 = \left(-\frac{3}{2}\right)^2 = \frac{9}{4}$$

x	$\langle -\infty, -2\rangle$	-2	$\langle -2, \frac{-1-\sqrt{3}}{2}\rangle$	$\frac{-1-\sqrt{3}}{2}$	$\langle \frac{-1-\sqrt{3}}{2}, -\frac{1}{2}\rangle$	$-\frac{1}{2}$
$f(x)$	$+$	0	$+$	$\frac{9}{4}$	$+$	$\frac{81}{16}$
$f'(x)$	$-$	0	$+$	$+$	$+$	0
$f''(x)$	$+$	$+$	$+$	0	$-$	$-$
	poz. sil konv.	nultočka i min.	poz. uzl. konv.	infl.	pozit. uzl. konk.	max

x	$\langle -\frac{1}{2}, \frac{-1+\sqrt{3}}{2}\rangle$	$\frac{-1+\sqrt{3}}{2}$	$\langle \frac{-1+\sqrt{3}}{2}, 1\rangle$	1	$\langle 1, +\infty\rangle$
$f(x)$	$+$	$\frac{9}{4}$	$+$	0	$+$
$f'(x)$	$-$	$-$	$-$	0	$+$
$f''(x)$	$-$	0	$+$	$+$	$+$
	pozit. sil. konk.	infl.	pozit. sil. konv.	nultočka i min.	poz. uzl. konv

1)



2) $f(x) = (x-1)^2(x+1)^2$,

$$D_f = \mathbf{R}, \quad \text{nema asimptota}$$

$$x_1, x_2 = -1 \quad \text{nultočka}$$

$$f(x) \geqslant 0, \quad \forall x \in \mathbf{R}$$

$$f(x) = (x^2-1)^2 = x^4 - 2x^2 + 1$$

$$f'(x) = 4x^3 - 4x = 4x(x^2-1)$$

$$f'(x) = 0 \implies x_1 = 0, \quad x_2 = -1, \quad x_3 = 1, \quad f(0) = 1$$

$$f''(x) = 12x^2 - 4 = 4(3x^2-1) = 4(x\sqrt{3}-1)(x\sqrt{3}+1)$$

$$f''(0) < 0, \quad f''(-1) = f''(1) > 0$$

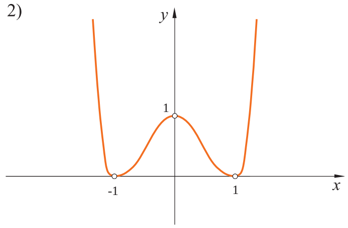
$$f''(x) = 0 \implies x_{1,2} = \pm\frac{\sqrt{3}}{3}$$

$$f\left(\frac{\sqrt{3}}{3}\right) = f\left(-\frac{\sqrt{3}}{3}\right) = \frac{4}{9}$$

x	$\langle -\infty, -1\rangle$	-1	$\langle -1, -\frac{\sqrt{3}}{3}\rangle$	$-\frac{\sqrt{3}}{3}$	$\langle -\frac{\sqrt{3}}{3}, 0\rangle$	0
$f(x)$	$+$	0	$+$	$\frac{4}{9}$	$+$	1
$f'(x)$	$-$	0	$+$	$+$	$+$	0
$f''(x)$	$+$	$+$	$+$	0	$-$	$-$
	pozit. sil. konv.	nultočka i min	pozit. uzl. konv.	infl.	pozit. uzl. konk.	max

x	$\langle 0, \frac{\sqrt{3}}{3} \rangle$	$\frac{\sqrt{3}}{3}$	$\langle \frac{\sqrt{3}}{3}, 1 \rangle$	1	$\langle 1, +\infty \rangle$
$f(x)$	$+$	$\frac{4}{9}$	$+$	0	$+$
$f'(x)$	$-$	$-$	$-$	0	$+$
$f''(x)$	$-$	$+$	$+$	$+$	
	pozit. sil. konk.	infl.	pozit. sil. konv.	nultočka i min.	pozit. uzl. konv.

2)



3) $f(x) = 3x^4 - 4x^3 + 1$,

$D_f = \mathbf{R}$, nema asimptota

$$3x^4 - 4x^3 + 1 = 0 \iff (x-1)^2 \overbrace{(3x^2 + 2x + 1)}^{D<0} = 0 \implies x_{1,2} = 1$$

	3	−4	0	0	1
1	3	−1	−1	−1	0
1	3	2	1	0	
1	3	5	6		

$$f'(x) = 12x^3 - 12x^2 = 12x^2(x-1)$$
$$f'(x) = 0 \implies x_{1,2} = 0, \quad x_3 = 1, \quad f(0) = 1$$
$$f''(x) = 36x^2 - 24x = 12x(3x-2)$$
$$f''(0) = 0, \quad f''(1) > 0$$
$$f''(x) = 0 \implies x_1 = 0, \quad x_2 = \frac{2}{3}, \quad f\left(\frac{2}{3}\right) = \frac{11}{27}$$

Kako odrediti karakter točke $(0, 1)$:

I. način

$$f'(0-\delta) < 0, \quad f'(0+\delta) < 0, \quad \delta \text{ malo} \implies \text{nije ekstrem}$$
$$f''(0-\delta) > 0, \quad f''(0+\delta) < 0, \quad \delta \text{ malo} \implies \text{infleksija}$$

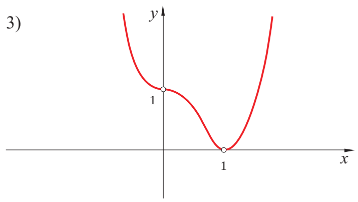
II. način

$$f'''(x) = 72x - 24, \quad f'''(0) \neq 0 \implies n = 3 \quad \text{infleksija}$$

Definicija. Neka je $f'(x_0) = f''(x_0) = \ldots = f^{(n-1)}(x_0) = 0, \quad f^{(n)}(x_0) \neq 0$. Tada $n = 2k \implies$ ekstrem $n = 2k+1 \implies$ infleksija

x	$\langle -\infty, 0\rangle$	0	$\left\langle 0, \frac{2}{3}\right\rangle$	$\frac{2}{3}$	$\left\langle \frac{2}{3}, 1\right\rangle$	1	$\langle 1+\infty\rangle$
$f(x)$	$+$	1	$+$	$\frac{11}{27}$	$+$	0	$+$
$f'(x)$	$-$	0	$-$	$-$	$-$	0	$+$
$f''(x)$	$+$	0	$-$	0	$+$	$+$	$+$
	pozit. silaz. konv.	infl.	pozit. silaz. konk.	infl.	pozit. silaz. konv.	nul-točka i min.	pozit. uzlaz. konv.

3)



4) $f(x) = x(x+1)(x-1)(x-2) = x^4 - 2x^3 - x^2 + 2x$,

$$D_f = \mathbf{R}$$

$$\lim_{x\to\pm\infty} f(x) = +\infty \implies \text{nema asimptota}$$

$$\lim_{x\to\pm\infty} \frac{f(x)}{x} = \pm\infty \implies \text{nema asimptota}$$

$$x_1 = 0, \quad x_2 = -1, \quad x_3 = 1, \quad x_4 = 2 \implies \text{nultočke}$$

$$\begin{array}{l}
(2x^3 - 3x^2 - x + 1) \quad : \left(x - \frac{1}{2}\right) = 2x^2 - 2x - 2 \\
\underline{-2x^3 \pm x^2} \\
\quad -2x^2 - x + 1 \\
\quad \underline{\pm 2x^2 \mp x} \\
\qquad -2x + 1 \\
\qquad \underline{\pm 2x \mp 1} \\
\qquad\qquad 0
\end{array}$$

$$f'(x) = 4x^3 - 6x^2 - 2x + 2$$

$$f'(x) = 0 = 2x^3 - 3x^2 - x + 1 = 0$$

$$2\left(x - \frac{1}{2}\right)(x^2 - x - 1) = 0$$

$$x_1 = \frac{1}{2}, \quad f\left(\frac{1}{2}\right) = \frac{15}{16}$$

$$x_{2,3} = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}, \quad f(x_{2,3}) \approx -\frac{1}{2}$$

$$f''(x) = 12x^2 - 12x - 2 = 2(6x^2 - 6x - 1)$$

$$f''(x) = 0 \implies x_{1,2} = \frac{3 \pm \sqrt{15}}{6}$$

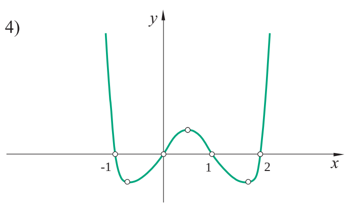
$$f''\left(\frac{1}{2}\right) < 0, \quad f''\left(\frac{1 \pm \sqrt{5}}{2}\right) > 0$$

x	$\langle -\infty, -1\rangle$	-1	$\langle -1, \frac{1-\sqrt{5}}{2}\rangle$	$\frac{1-\sqrt{5}}{2}$	$\langle \frac{1-\sqrt{5}}{2}, \frac{3-\sqrt{15}}{6}\rangle$	$\frac{3-\sqrt{15}}{6}$
$f(x)$	$+$	0	$-$	$-$	$-$	$-$
$f'(x)$	$-$	$-$	$-$	0	$+$	$+$
$f''(x)$	$+$	$+$	$+$	$+$	$+$	0
	poz. sil. konv.	nul-točka	neg. sil. konv.	min	neg. uzl. konv.	infl.

x	$\langle \frac{3-\sqrt{15}}{6}, 0\rangle$	0	$\langle 0, \frac{1}{2}\rangle$	$\frac{1}{2}$	$\langle \frac{1}{2}, 1\rangle$	1	$\langle 1, \frac{3+\sqrt{15}}{6}\rangle$
$f(x)$	$-$	0	$+$	$\frac{15}{16}$	$+$	0	$-$
$f'(x)$	$+$	$+$	$+$	0	$-$	$-$	$-$
$f''(x)$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
	neg. uzl. konk.	nul-točka.	poz. uzl. konk.	max	poz. sil. konk.	nul-točka	neg. sil. konk.

x	$\frac{3+\sqrt{15}}{6}$	$\langle \frac{3+\sqrt{15}}{6}, \frac{1+\sqrt{5}}{2}\rangle$	$\frac{1+\sqrt{5}}{2}$	$\langle \frac{1+\sqrt{5}}{2}, 2\rangle$	2	$\langle 2, +\infty\rangle$
$f(x)$	$-$	$-$	$-$	$-$	0	$+$
$f'(x)$	$-$	$-$	0	$+$	$+$	$+$
$f''(x)$	0	$+$	$+$	$+$	$+$	$+$
	infl.	neg. sil. konv.	min	neg. uzl. konv.	nul-točka	poz. uzl. konv.

4)



5) $f(x) = x^4 + x$,

$$D_f = \mathbf{R}, \quad \lim_{x\to\pm\infty} f(x) = +\infty, \quad \lim_{x\to\pm\infty} \frac{f(x)}{x} = \pm\infty \implies \text{nema asimptota}$$

$$x^4 + x = 0 \implies x(x^3+1) = 0 \implies x_1 = 0, \quad x_2 = -1 \quad \text{– nultočke}$$

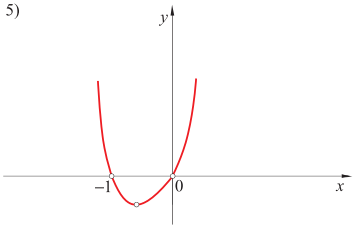
x	$\langle -\infty, -1\rangle$	$\langle -1, 0\rangle$	$\langle 0, +\infty\rangle$
$f(x)$	$+$	$-$	$+$

$$f'(x) = 4x^3 + 1$$

$$4x^3 + 1 = 0 \implies x^3 = -\frac{1}{4} \implies x = \sqrt[3]{-\frac{1}{4}} = -\frac{1}{\sqrt[3]{4}} \approx -0.63$$

x	$\left\langle -\infty, \sqrt[3]{-\frac{1}{4}} \right\rangle$	$\left\langle \sqrt[3]{-\frac{1}{4}}, +\infty \right\rangle$
$f'(x)$	$-$	$+$

$$f'''(x) = 12x^2 \geqslant 0, \quad \forall x \in \mathbf{R}$$
$$f''\left(\sqrt[3]{-\frac{1}{4}}\right) > 0 \implies \text{min}$$
$$f''(x) = 0 \implies x = 0$$



6) $f(x) = x^4 - 4x^2$,

$$D_f = \mathbf{R}, \quad \lim_{x\to\pm\infty} f(x) = \infty, \quad \lim_{x\to\pm\infty} \frac{f(x)}{x} = \infty \implies \text{nema asimptota}$$
$$x^4 - 4x^2 = x^2(x-2)(x+2) \implies x_1 = -2, \quad x_2 = 0, \quad x_3 = 2 \quad \text{– nultočke}$$

x	$\langle -\infty, -2\rangle$	$\langle -2, 0\rangle$	$\langle 0, 2\rangle$	$\langle 2, +\infty\rangle$
$f(x)$	$+$	$-$	$-$	$+$

$$f'(x) = 4x^3 - 8x$$
$$f'(x) = 0 \implies 4x^3 - 8x = 0 \implies 4x(x-\sqrt{2})(x+\sqrt{2}) = 0$$
$$\implies x = 0, \quad x_1 = \sqrt{2}, \quad x = -\sqrt{2}$$

x	$\langle -\infty, -\sqrt{2}\rangle$	$\langle -\sqrt{2}, 0\rangle$	$\langle 0, \sqrt{2}\rangle$	$\langle \sqrt{2}, +\infty\rangle$
$f'(x)$	$-$	$+$	$-$	$+$

$$f''(x) = 12x^2 - 8$$
$$f''(0) = -8 < 0, \quad f(0) = 0 \implies (0,0) \quad \text{max}$$
$$f''(\sqrt{2}) = 16 > 0, \quad f(\sqrt{2}) = -4 \implies (\sqrt{2}, -4) \quad \text{min}$$
$$f''(-\sqrt{2}) = 16 > 0, \quad f(-\sqrt{2}) = -4 \implies (-\sqrt{2}, -4) \quad \text{min}$$
$$12x^2 - 8 = 0 \implies x^2 = \frac{2}{3} \implies x \pm \sqrt{\frac{2}{3}} \approx x \pm 0.8 \quad \text{– infleksija}$$

6)