Zadatak 3. Grafički prikaži sljedeće funkcije:

1) $f(x)=\dfrac{1}{x^2-4}$; **2)** $f(x)=\dfrac{x^2}{x^2-4}$;

3) $f(x)=\dfrac{x}{x^2-4}$; **4)** $f(x)=\dfrac{x+1}{x^3}$;

5) $f(x)=\dfrac{x-1}{x^2(x-2)}$; **6)** $f(x)=\dfrac{x^2-2x+1}{x^2+1}$;

7) $f(x)=\dfrac{x}{x^2+1}$; **8)** $f(x)=\dfrac{x^2-2x}{x+1}$.

Rješenje. **1)** $f(x)=\dfrac{1}{x^2-4}$,

$D_f=\mathbf{R}\setminus\{-2,2\}$

$$\left.\begin{aligned}&\lim_{x\to-2^-}\frac{1}{x^2-4}=+\infty,\quad \lim_{x\to-2^+}\frac{1}{x^2-4}=-\infty\\&\lim_{x\to2^-}\frac{1}{x^2-4}=-\infty,\quad \lim_{x\to2^+}\frac{1}{x^2-4}=+\infty\end{aligned}\right\}\implies x=\pm2 \text{ vertikalna asimptota}$$

$$\lim_{x\to\pm\infty}\frac{1}{x^2-4}=0\implies y=0\quad\text{horizontalna asimptota}$$

x	$\langle-\infty,-2\rangle$	$\langle-2,2\rangle$	$\langle2,+\infty\rangle$
$f(x)$	$+$	$-$	$+$

$$f(x)=(x^2-4)^{-1}$$

$$f'(x)=-(x^2-4)^{-2}\cdot 2x=-\frac{2x}{(x^2-4)^2}$$

$$f'(x)=0\implies x=0,\quad f(0)=-\frac{1}{4}$$

x	$\langle-\infty,-2\rangle$	$\langle-2,0\rangle$	$\langle0,2\rangle$	$\langle2,+\infty\rangle$
$f'(x)$	$+$	$+$	$-$	$-$

$$f'(x)=-2x(x^2-4)^{-2}$$

$$f''(x)=-2(x^2-4)^{-2}+(-2x)\cdot(-2)(x^2-4)^{-3}\cdot 2x$$

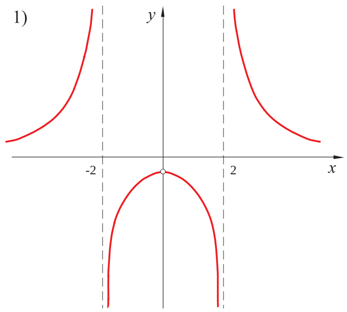
$$=-\frac{2}{(x^2-4)^2}+\frac{8x^2}{(x^2-4)^3}=\frac{8x^2-2x^2+8}{(x^2-4)^3}=\frac{6x^2+8}{(x^2-4)^3}$$

$$f''(0)<0\implies M\left(0,-\frac{1}{4}\right),\quad f''(x)\neq0,\quad \forall x\in D_f$$

x	$\langle-\infty,-2\rangle$	$\langle-2,2\rangle$	$\langle2,+\infty\rangle$
$f''(x)$	$+$	$-$	$+$

x	$\langle-\infty,-2\rangle$	-2	$\langle-2,0\rangle$	0	$\langle0,2\rangle$	2	$\langle2,+\infty\rangle$
$f(x)$	$+$	N.E.	$-$	$-\frac{1}{4}$	$-$	N.E.	$+$
$f'(x)$	$+$	N.E.	$+$	0	$-$	N.E.	$-$
$f''(x)$	$+$	N.E.	$-$	$-$	$-$	N.E.	$+$

1)



2) $f(x)=\dfrac{x^2}{x^2-4}$,

$D_f=\mathbf{R}\setminus\{-2,2\}$

$$\left.\begin{array}{l}\lim\limits_{x\to-2^-}\dfrac{x^2}{x^2-4}=+\infty, \quad \lim\limits_{x\to-2^+}\dfrac{x^2}{x^2-4}=-\infty \\ \lim\limits_{x\to2^-}\dfrac{x^2}{x^2-4}=-\infty, \quad \lim\limits_{x\to2^+}\dfrac{x^2}{x^2-4}=+\infty\end{array}\right\} \implies x=\pm 2 \text{ vertikalna asimptota}$$

$$\lim_{x\to\pm\infty}\frac{x^2}{x^2-4}=1 \implies y=1 \quad \text{horizontalna asimptota}$$

$$f(x)=0 \implies x^2=0 \implies x=0 \quad \text{nultočka}$$

x	$\langle-\infty,-2\rangle$	$\langle-2,0\rangle$	$\langle0,2\rangle$	$\langle2,+\infty\rangle$
$f(x)$	$+$	$-$	$-$	$+$

$$f'(x)=\frac{2x(x^2-4)-2x\cdot x^2}{(x^2-4)^2}=\frac{2x^3-8x-2x^3}{(x^2-4)^2}=-\frac{8x}{(x^2-4)^2}$$

$$f'(x)=0 \implies x=0$$

x	$\langle-\infty,-2\rangle$	$\langle-2,0\rangle$	$\langle0,2\rangle$	$\langle2,+\infty\rangle$
$f'(x)$	$+$	$+$	$-$	$-$

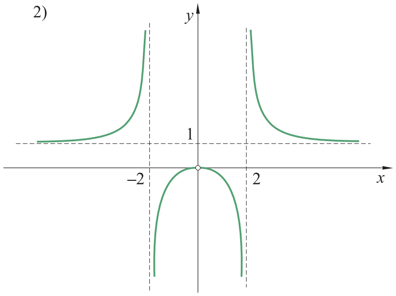
$$f''(x)=-\left[\frac{8(x^2-4)^2-8x\cdot2(x^2-4)\cdot2x}{(x^2-4)^4}\right]=-8\frac{x^2-4-4x^2}{(x^2-4)^3}=\frac{8(3x^2+4)}{(x^2-4)^3}$$

$$f''(0)<0 \implies M(0,0), \quad f''(x)\neq0, \quad \forall x\in D_f$$

x	$\langle-\infty,-2\rangle$	$\langle-2,2\rangle$	$\langle2,+\infty\rangle$
$f''(x)$	$+$	$-$	$+$

x	$\langle-\infty,-2\rangle$	-2	$\langle-2,0\rangle$	0	$\langle0,2\rangle$	2	$\langle2,+\infty\rangle$
$f(x)$	$+$	N.E.	$-$	0	$-$	N.E.	$+$
$f'(x)$	$+$	N.E.	$+$	0	$-$	N.E.	$-$
$f''(x)$	$+$	N.E.	$-$	$-$	$-$	N.E.	$+$

2)



3) $f(x) = \dfrac{x}{x^2-4}$,

$D_f = \mathbf{R} \setminus \{-2, 2\}$

$f(x) = 0 \implies x = 0$ nultočka

$$\left.\begin{array}{ll} \lim\limits_{x\to -2^-} \dfrac{x}{x^2-4} = -\infty, & \lim\limits_{x\to -2^+} \dfrac{x}{x^2-4} = +\infty \\ \lim\limits_{x\to 2^-} \dfrac{x}{x^2-4} = -\infty, & \lim\limits_{x\to 2^+} \dfrac{x}{x^2-4} = +\infty \end{array}\right\} \implies x = \pm 2 \quad \text{vertikalna asimptota}$$

$$\lim_{x\to\pm\infty} \frac{x}{x^2-4} = 0 \implies y = 0 \quad \text{horizontalna asimptota}$$

x	$\langle -\infty, -2\rangle$	$\langle -2, 0\rangle$	$\langle 0, 2\rangle$	$\langle 2, +\infty\rangle$
$f(x)$	$-$	$+$	$-$	$+$

$$f'(x) = \frac{x^2-4-x\cdot 2x}{(x^2-4)^2} = -\frac{x^2+4}{(x^2-4)^2}, \quad f'(x) \neq 0, \quad \forall x \in D_f$$

x	$\langle -\infty, -2\rangle$	$\langle -2, 2\rangle$	$\langle 2, +\infty\rangle$
$f'(x)$	$-$	$-$	$-$

$$f''(x) = -\frac{2x(x^2-4)^2-(x^2+4)2(x^2-4)2x}{(x^2-4)^3} = -\frac{2x^3-8x-4x^3-16x}{(x^2-4)^3}$$

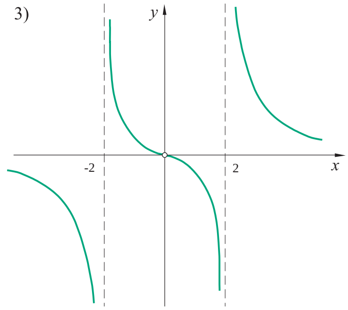
$$f''(x) = \frac{2x^3+24x}{(x^2-4)^3} = \frac{2x(x^2+12)}{(x^2-4)^3}$$

$$f''(x) = 0 \implies x = 0 \quad \text{infleksija}$$

x	$\langle -\infty, -2\rangle$	$\langle -2, 0\rangle$	$\langle 0, 2\rangle$	$\langle 2, +\infty\rangle$
$f''(x)$	$-$	$+$	$-$	$+$

x	$\langle -\infty, -2\rangle$	-2	$\langle -2, 0\rangle$	0	$\langle 0, 2\rangle$	2	$\langle 2, +\infty\rangle$
$f(x)$	$-$	N.E.	$+$	0	$-$	N.E.	$+$
$f'(x)$	$-$	N.E.	$-$	$-$	$-$	N.E.	$-$
$f''(x)$	$-$	N.E.	$+$	0	$-$	N.E.	$+$

3)



4) $f(x)=\dfrac{x+1}{x^3}$,

$D_f=\mathbf{R}\setminus\{0\}$

$n=-1$ nultočka

$\lim\limits_{x\to 0^-}\dfrac{x+1}{x^3}=-\infty, \quad \lim\limits_{x\to 0^+}\dfrac{x+1}{x^3}=+\infty, \quad x=0$ vertikalna asimptota

$\lim\limits_{x\to\pm\infty}\dfrac{x+1}{x^3}=0 \implies y=0$ horizontalna asimptota

x	$\langle -\infty,-1\rangle$	$\langle -1,0\rangle$	$\langle 0,+\infty\rangle$
$f(x)$	$+$	$-$	$+$

$$f'(x)=\frac{x^3-3x^2(x+1)}{x^6}=\frac{x^3-3x^3-3x^2}{x^6}=\frac{-2x^3-3x^2}{x^6}=\frac{-2x-3}{x^4}$$

$$f'(x)=0 \implies x=-\frac{3}{2}, \quad f\left(-\frac{3}{2}\right)=\frac{4}{27}$$

x	$\left\langle -\infty,-\frac{3}{2}\right\rangle$	$\left\langle -\frac{3}{2},0\right\rangle$	$\langle 0,+\infty\rangle$
$f'(x)$	$+$	$-$	$-$

$$f''(x)=\frac{-2x^4+(2x+3)4x^3}{x^8}=\frac{x^3(-2x+8x+12)}{x^8}=\frac{6(x+2)}{x^5}$$

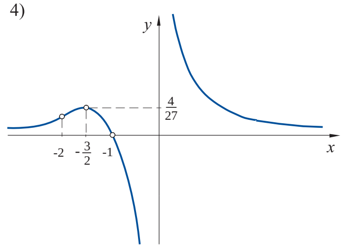
$$f''\left(-\frac{3}{2}\right)<0 \implies M\left(-\frac{3}{2},\frac{4}{27}\right)$$

$$f''(x)=0 \implies x=-2, \quad f(-2)=\frac{1}{8} \implies \left(-2,\frac{1}{8}\right) \quad \text{infleksija}$$

x	$\langle -\infty,-2\rangle$	$\langle -2,0\rangle$	$\langle 0,+\infty\rangle$
$f''(x)$	$+$	$-$	$+$

x	$\langle -\infty,-2\rangle$	-2	$\left\langle -2,-\frac{3}{2}\right\rangle$	$-\frac{3}{2}$	$\left\langle -\frac{3}{2},-1\right\rangle$	-1	$\langle -1,0\rangle$	0	$\langle 0,+\infty\rangle$
$f(x)$	$+$	$\frac{1}{8}$	$+$	$\frac{4}{27}$	$+$	0	$-$	N.E.	$+$
$f'(x)$	$+$	$+$	$+$	0	$-$	$-$	$-$	N.E.	$-$
$f''(x)$	$+$	0	$-$	$-$	$-$	$-$	$-$	N.E.	$+$

4)



5) $f(x)=\dfrac{x-1}{x^2(x-2)}$,

$D_f=\mathbf{R}\setminus\{0,2\}$

$f(x)=0 \implies x=1$ nultočka

$\lim\limits_{x\to 0^-}\dfrac{x-1}{x^2(x-2)}=+\infty,\quad \lim\limits_{x\to 0^+}\dfrac{x-1}{x^2(x-2)}=+\infty,\quad x=0$ vertikalna asimptota

$\lim\limits_{x\to 2^-}\dfrac{x-1}{x^2(x-2)}=-\infty,\quad \lim\limits_{x\to 2^+}=\dfrac{x-1}{x^2(x-2)}=+\infty,\quad x=2$ vertikalna asimptota

$\lim\limits_{x\to\pm\infty}f(x)=0 \implies y=0$ horizontalna asimptota

x	$\langle -\infty,0\rangle$	$\langle 0,1\rangle$	$\langle 1,2\rangle$	$\langle 2,+\infty\rangle$
$f(x)$	$+$	$+$	$-$	$+$

$$f(x)=\frac{x-1}{x^3-2x^2}$$

$$f'(x)=\frac{x^3-2x^2-(x-1)(3x^2-4x)}{x^4(x-2)^2}=\frac{x^3-2x^2-(3x^3-4x^2-3x^2+4x)}{x^4(x-2)^2}$$

$$=\frac{x^3-2x^2-3x^3+7x^2-4x}{x^4(x-2)^2}=\frac{-2x^3+5x^2-4x}{x^4(x-2)^2}=\frac{-x(2x^2-5x+4)}{x^4(x-2)^2}$$

$$=-\frac{2x^2-5x+4}{x^3(x-2)^2}$$

$f'(x)\neq 0,\quad \forall x\in D_f$

x	$\langle -\infty,0\rangle$	$\langle 0,2\rangle$	$\langle 2,+\infty\rangle$
$f'(x)$	$+$	$-$	$-$

$$f'(x)=\frac{-2x^2+5x-4}{x^3(x-2)^2}$$

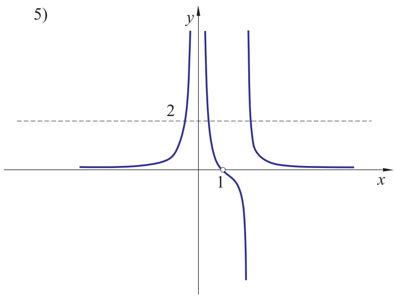
$$f''(x)=\frac{(-4x+5)(x-2)^2x^3+(2x^2-5x+4)3x^2 2(x-2)}{x^6(x-2)^4}$$

$$=\frac{x(x-2)(5-4x)+6(2x^2-5x+4)}{x^4(x-2)^3}=\frac{(x^2-2x)(5-4x)+12x^2-30x+24}{x^4(x-2)^3}$$

$$=\frac{-4x^3+13x^2-10x+12x^2-30x+24}{x^4(x-2)^3}=\frac{-4x^3+25x^2-40x+24}{x^4(x-2)^3}$$

x	$\langle -\infty, 0\rangle$	$\langle 0, 2\rangle$	$\langle 2, +\infty\rangle$
$f''(x)$	$+$	$+/-$	$+$

5)



6) $f(x) = \dfrac{x^2 - 2x + 1}{x^2 + 1} = \dfrac{(x-1)^2}{x^2+1}$,

$D_f = \mathbf{R}$

$\lim\limits_{x\to\pm\infty} \dfrac{x^2 - 2x + 1}{x^2 + 1} = 1 \implies y = 1$ horizontalna asimptota

$f(x) = 0 \implies x = 1$ nultočka

x	$\langle -\infty, 1\rangle$	$\langle 1, +\infty\rangle$
$f(x)$	$+$	$+$

$$f'(x) = \frac{(2x-2)(x^2+1) - (x^2-2x+1)\cdot 2x}{(x^2+1)^2} = \frac{2x^3 - 2x^2 + 2x - 2 - 2x^3 + 4x^2 - 2x}{(x^2+1)^2}$$

$$f'(x) = \frac{2x^2 - 2}{(x^2+1)^2}$$

$$f'(x) = 0 \implies x = \pm 1, \quad f(1) = 0, \quad f(-1) = 2$$

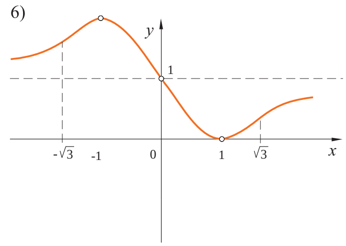
$$f''(x) = \frac{4x(x^2+1)^2 - (2x^2-2)2\cdot(x^2+1)\cdot 2x}{(x^2+1)^4} = \frac{4x^3 + 4x - 8x^3 + 8x}{(x^2+1)^3} = \frac{12x - 4x^3}{(x^2+1)^3}$$

$$f''(x) = \frac{4x(3-x^2)}{(x^2+1)^3}$$

$$f''(1) > 0 \implies m(1,0), \quad f''(-1) < 0 \implies M(-1,2)$$

$$f''(x) = 0 \implies x_1 = 0,\ x_{2,3} = \pm\sqrt{3},\ f(0) = 1,\ f(-\sqrt{3}) = 1.866,\ f(\sqrt{3}) = 0.134$$

x	$\langle -\infty, -\sqrt{3}\rangle$	$\langle -\sqrt{3}, 0\rangle$	$\langle 0, \sqrt{3}\rangle$	$\langle \sqrt{3}, +\infty\rangle$
$f''(x)$	$+$	$-$	$+$	$-$



7) $f(x)=\dfrac{x}{x^2+1}$,

$$D_f=\mathbf{R},\quad f(x)=0 \implies x=0 \quad \text{nultočka}$$

$$\lim_{x\to\pm\infty} f(x)=0 \implies y=0 \quad \text{horizontalna asimptota}$$

x	$\langle -\infty,0\rangle$	$\langle 0,+\infty\rangle$
$f(x)$	$-$	$+$

$$f'(x)=\frac{x^2+1-x\cdot 2x}{(x^2+1)^2}=\frac{x^2+1-2x^2}{(x^2+1)^2}=\frac{1-x^2}{(1+x^2)^2}$$

$$f'(x)=0 \implies x=\pm 1,\quad f(-1)=-\frac{1}{2},\quad f(1)=\frac{1}{2}$$

x	$\langle -\infty,-1\rangle$	$\langle -1,1\rangle$	$\langle 1,+\infty\rangle$
$f(x)$	$-$	$+$	$-$

$$f''(x)=\frac{-2x(1+x^2)^2+(x^2-1)\cdot 2(1+x^2)\cdot 2x}{(1+x^2)^4}=\frac{-2x(1+x^2)+x(x^2-1)}{(1+x^2)^3}$$

$$=\frac{-2x-2x^3+4x^3-4x}{(1+x^2)^3}=\frac{2x^3-6x}{(1+x^2)^3}=\frac{2x(x-\sqrt{3})(x+\sqrt{3})}{(1+x^2)^3}$$

$$f''(-1)>0 \implies m\left(-1,-\frac{1}{2}\right)$$

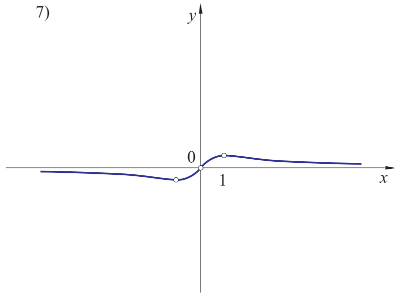
$$f''(1)<0 \implies M\left(1,\frac{1}{2}\right)$$

$$f''(x)=0 \implies x=0,\pm\sqrt{3},\ f(0)=0,\ f(-\sqrt{3})=-\frac{\sqrt{3}}{4},\ f(\sqrt{3})=\frac{\sqrt{3}}{4}$$

$$\implies (0,0),\quad \left(-\sqrt{3},-\frac{\sqrt{3}}{4}\right),\quad \left(\sqrt{3},\frac{\sqrt{3}}{4}\right) \quad \text{infleksija}$$

x	$\langle -\infty,-\sqrt{3}\rangle$	$\langle -\sqrt{3},0\rangle$	$\langle 0,\sqrt{3}\rangle$	$\langle \sqrt{3},+\infty\rangle$
$f''(x)$	$-$	$+$	$-$	$+$

7)



8) $f(x) = \dfrac{x^2 - 2x}{x+1}$,

$D_f = \mathbf{R} \setminus \{-1\}$

$\lim\limits_{x \to -1^-} \dfrac{x^2-2x}{x+1} = -\infty, \quad \lim\limits_{x \to -1^+} \dfrac{x^2-2x}{x+1} = +\infty, \quad x = -1$ vertikalna asimptota

$f(x) = 0 \implies x_1 = 0, \quad x_2 = 2$ nultočka

$\lim\limits_{x \to \pm\infty} \dfrac{x^2-2x}{x+1} = \infty, \quad \lim\limits_{x \to \pm\infty} \dfrac{f(x)}{x} = \lim\limits_{x \to \pm\infty} \dfrac{x^2-2x}{x^2+x} = 1$

$\lim\limits_{x \to \pm\infty} (f(x) - kx) = \lim\limits_{x \to \pm\infty} \left(\dfrac{x^2-2x}{x+1} - x \right) = \lim\limits_{x \to \pm\infty} \dfrac{x^2 - 2x - x^2 - x}{x+1} = -3$

$\implies y = x - 3$ kosa asimptota

x	$\langle -\infty, -1 \rangle$	$\langle -1, 0 \rangle$	$\langle 0, 2 \rangle$	$\langle 2, +\infty \rangle$
$f(x)$	$-$	$+$	$-$	$+$

$$f'(x) = \frac{(2x-2)(x+1) - (x^2-2x)}{(x+1)^2} = \frac{2x^2 - 2x + 2x - 2 - x^2 + 2x}{(x+1)^2} = \frac{x^2+2x-2}{(x+1)^2}$$

$$f'(x) = \frac{x^2+2x+1-3}{(x+1)^2} = 1 - \frac{3}{(x+1)^2} = 1 - 3(x+1)^{-2}$$

$$f'(x) = 0 \implies x^2 + 2x - 2 = 0 \implies x^2 + 2x + 1 = 3$$

$$\implies (x+1)^2 = 3 \implies x + 1 = \pm\sqrt{3}$$

$$\implies x_{1,2} = -1 \pm \sqrt{3}, \quad f(x_1) = -7.5, \quad f(x_2) = -0.5$$

x	$\langle -\infty, -1-\sqrt{3} \rangle$	$\langle -1-\sqrt{3}, -1 \rangle$	$\langle -1, -1+\sqrt{3} \rangle$	$\langle -1+\sqrt{3}, +\infty \rangle$
$f'(x)$	$+$	$-$	$-$	$+$

$$f''(x) = \frac{6}{(x+1)^3}, \quad f''(-1-\sqrt{3}) < 0 \implies M(-1-\sqrt{3}, -7.5)$$

$$f''(-1+\sqrt{3}) > 0 \implies m(-1+\sqrt{3}, -0.5)$$

$$f''(x) \neq 0, \quad \forall x \in D_f$$

x	$\langle -\infty, -1 \rangle$	$\langle -1, +\infty \rangle$
$f''(x)$	$-$	$+$

8)