Zadatak 4. Grafički prikaži sljedeće funkcije:

1) $f(x) = 5\dfrac{x-2}{x^2}$;

2) $f(x) = \dfrac{x^2-1}{x^2+1}$;

3) $f(x) = \dfrac{x^2-2x+4}{x^2+x-2}$;

4) $f(x) = \dfrac{x^2-4x+3}{x^2-2x}$;

5) $f(x) = \dfrac{x^2-1}{x^4+1}$;

6) $f(x) = \dfrac{1-4x^2}{4x^2(1+x^2)}$;

7) $f(x) = \dfrac{1}{x+1} + \dfrac{1}{x-1}$;

8) $f(x) = \dfrac{1-x}{x-2}$.

Rješenje. **1)** $f(x) = \dfrac{5(x-2)}{x^2}$

$D_f = \mathbf{R} \setminus \{0\}$

$\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^-} f(x) = -\infty, \quad x = 0$ vertikalna asimptota

$f(x) = 0 \implies x = 2$ nultočka

$\lim\limits_{x\to \pm\infty} f(x) = 0, \quad y = 0$ horizontalna asimptota

x	$\langle -\infty, 0\rangle$	$\langle 0, 2\rangle$	$\langle 2, +\infty\rangle$
$f(x)$	$-$	$-$	$+$

$$f'(x) = 5\frac{x^2-(x-2)2x}{x^4} = 5\frac{x-2x+4}{x^3} = \frac{5(4-x)}{x^3}$$

$$f'(x) = 0 \implies x = 4, \quad f(4) = \frac{5}{8}$$

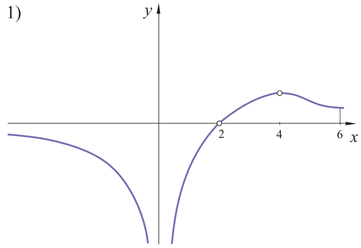
x	$\langle -\infty, 0\rangle$	$\langle 0, 4\rangle$	$\langle 4, +\infty\rangle$
$f'(x)$	$-$	$+$	$-$

$$f''(x) = 5\frac{(-1)x^3-(4-x)3x^2}{x^6} = 5\frac{-x-(4-x)3}{x^4}$$
$$= \frac{5(-x-12+3x)}{x^4} = \frac{5(2x-12)}{x^4} = \frac{10(x-6)}{x^4}$$

$$f''(4) < 0 \implies M\left(4, \frac{5}{8}\right)$$

$$f''(x) = 0 \implies x = 6, \quad f(6) = \frac{5}{9} \implies \left(6, \frac{5}{9}\right) \text{ infleksija}$$

x	$\langle -\infty, 0\rangle$	$\langle 0, 6\rangle$	$\langle 6, +\infty\rangle$
$f''(x)$	$-$	$-$	$+$



2) $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$,

$D_f = \mathbf{R}$

$f(x) = 0 \implies x = \pm 1$ nultočka

$\lim\limits_{x \to \pm\infty} \dfrac{x^2 - 1}{x^2 + 1} = 1 \implies y = 1$ horizontalna asimptota

x	$\langle -\infty, -1 \rangle$	$\langle -1, 1 \rangle$	$\langle 1, +\infty \rangle$
$f(x)$	$+$	$-$	$+$

$$f'(x) = \frac{2x(x^2+1) - (x^2-1)2x}{(x^2+1)^2} = \frac{2x(x^2+1-x^2+1)}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2}$$

$f'(x) = 0 \implies x = 0, \quad f(0) = -1$

x	$\langle -\infty, 0 \rangle$	$\langle 0, +\infty \rangle$
$f'(x)$	$-$	$+$

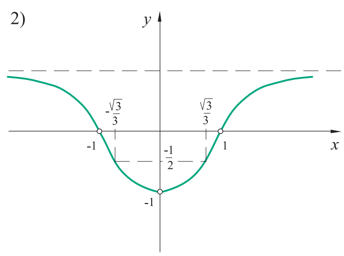
$$f''(x) = \frac{4(x^2+1)^2 - 4x \cdot 2 \cdot 2x(x^2+1)}{(x^2+1)^4} = \frac{4x^2 + 4 - 16x^2}{(x^2+1)^3} = \frac{4(1-3x^2)}{(x^2+1)^3}$$

$f''(0) > 0 \implies m(0, -1)$

$$f''(x) = 0 \implies x_{1,2} = \pm\frac{\sqrt{3}}{3}, \quad f\left(\pm\frac{\sqrt{3}}{3}\right) = \frac{\frac{3}{9} - 1}{\frac{3}{9} + 1} = \frac{-\frac{2}{3}}{\frac{4}{3}} = -\frac{1}{2}$$

$\implies \left(\pm\dfrac{\sqrt{3}}{3}, -\dfrac{1}{2}\right)$ infleksija

x	$\left\langle -\infty, -\frac{\sqrt{3}}{3} \right\rangle$	$\left\langle -\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3} \right\rangle$	$\left\langle \frac{\sqrt{3}}{3}, +\infty \right\rangle$
$f''(x)$	$-$	$+$	$-$



3) $f(x) = \dfrac{x^2 - 2x + 4}{x^2 + x - 2} = \dfrac{(x-1)^2 + 3}{(x-1)(x+2)}$,

$D_f = \mathbf{R} \setminus \{-2, 1\}$

$\lim\limits_{x\to -2^-} f(x) = +\infty, \quad \lim\limits_{x\to -2^+} f(x) = -\infty \quad x = -2$ vertikalna asimptota

$\lim\limits_{x\to 1^-} f(x) = -\infty, \quad \lim\limits_{x\to 1^+} f(x) = +\infty \quad x = 1$ vertikalna asimptota

$f(x) \neq 0, \quad \forall x \in \mathbf{R}, \quad \lim\limits_{x\to \pm\infty} f(x) = 1 \implies y = 1$ horizontalna asimptota

x	$\langle -\infty, -2\rangle$	$\langle -2, 1\rangle$	$\langle 1, +\infty\rangle$
$f(x)$	$+$	$-$	$+$

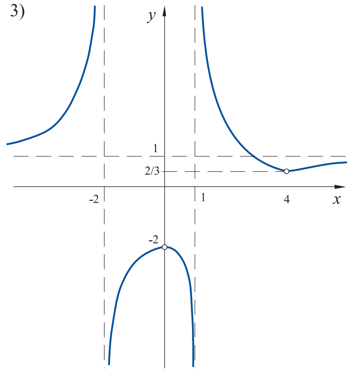
$$f'(x) = \frac{(2x-2)(x^2+x-2) - (2x+1)(x^2-2x+4)}{(x^2+x-2)^2}$$
$$= \frac{2x^3 + 2x^2 - 2x^2 - 4x - 2x + 4 - 2x^3 + 4x^2 - x^2 - 8x + 2x - 4}{(x^2+x-2)^2}$$
$$f'(x) = \frac{3x^2 - 12x}{(x^2+x-2)^2} = \frac{3x(x-4)}{(x^2+x-2)^2}$$
$$f'(x) = 0 \implies x_1 = 0, \quad x_2 = 4, \quad f(0) = -2, \quad f(4) = \frac{2}{3}$$

x	$\langle -\infty, -2\rangle$	$\langle -2, 0\rangle$	$\langle 0, 1\rangle$	$\langle 1, 4\rangle$	$\langle 4, +\infty\rangle$
$f'(x)$	$+$	$+$	$-$	$-$	$+$

$$f''(x) = \frac{(6x-12)(x^2+x-2)^2 - (3x^2-12x)2(x^2+x-2)(2x+1)}{(x^2+x-2)^4}$$
$$= \frac{(6x-12)(x^2+x-2) - (6x^2-24x)(2x+1)}{(x^2+x-2)^3}$$
$$= \frac{6x^3 + 6x^2 - 12x - 12x^2 - 12x + 24 - 12x^3 - 6x^2 + 48x^2 + 24x}{(x^2+x-2)^3}$$
$$= \frac{-6x^3 + 36x^2 + 24}{(x^2+x-2)^3}$$

$f''(0) < 0 \implies M(0, -2), \quad f''(4) < 0 \implies m\left(4, \dfrac{2}{3}\right)$ infleksiju teško naći

3)



4) $f(x)=\dfrac{x^2-4x+3}{x^2-2x}=\dfrac{(x-1)(x-3)}{x(x-2)}$,

$D_f=\mathbf{R}\setminus\{0,2\}$

$\lim\limits_{x\to 0^-} f(x)=+\infty,\quad \lim\limits_{x\to 0^+} f(x)=-\infty\quad x=0$ vertikalna asimptota

$\lim\limits_{x\to 2^-} f(x)=+\infty,\quad \lim\limits_{x\to 2^+} f(x)=-\infty,\quad x=2$ vertikalna asimptota

$\lim\limits_{x\to \pm\infty} f(x)=1 \implies y=1$ horizontalna asimptota

$f(x)=0 \implies x_1=1,\quad x_2=3$ nultočka

x	$\langle -\infty,0\rangle$	$\langle 0,1\rangle$	$\langle 1,2\rangle$	$\langle 2,3\rangle$	$\langle 3,+\infty\rangle$
$f(x)$	$+$	$-$	$+$	$-$	$+$

$$f'(x)=\frac{(2x-4)(x^2-2x)-(2x-2)(x^2-4x+3)}{(x^2-2x)^2}$$

$$=\frac{2x^3-4x^2-4x^2+8x-2x^3+8x^2-6x+2x^2-8x+6}{(x^2-2x)^2}$$

$$f'(x)=\frac{2x^2-6x+6}{(x^2-2x)^2}=\frac{2(x^2-3x+3)}{(x^2-2x)^2}>0,\quad \forall x\in D_f$$

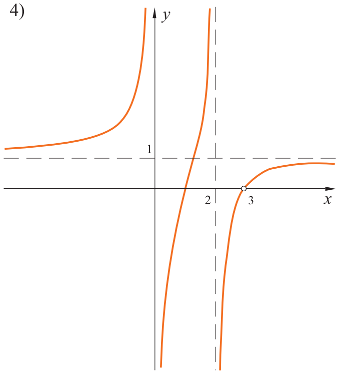
$$f''(x)=\frac{(4x-6)(x^2-2x)^2-2(x^2-2x)(2x-2)(2x^2-6x+6)}{(x^2-2x)^4}$$

$$=\frac{(4x-6)(x^2-2x)-(4x-4)(2x^2-6x+6)}{(x^2-2x)^3}$$

$$=\frac{4x^3-8x^2-6x^2+12x-8x^3+24x^2-24x+8x^2-24x+24}{(x^2-2x)^3}$$

$$f''(x)=\frac{-4x^3+18x^2-36x+24}{(x^2-2x)^3}$$ infleksiju teško naći

4)



5) $f(x)=\dfrac{x^2-1}{x^4+1}$,

$$D_f=\mathbf{R},\quad f(x)=0 \implies x_1=-1, x_2=1 \quad \text{nultočka}$$

$$\lim_{x\to\pm\infty} f(x)=0 \implies y=0 \quad \text{horizontalna asimptota}$$

x	$\langle -\infty,-1\rangle$	$\langle -1,1\rangle$	$\langle 1,+\infty\rangle$
$f(x)$	$+$	$-$	$+$

$$f'(x)=\frac{2x(x^4+1)-4x^3(x^2-1)}{(x^4+1)^2}=\frac{2x^5+2x-4x^5+4x^3}{(x^4+1)^2}$$

$$=\frac{-2x^5+4x^3+2x}{(x^4+1)^2}=\frac{-2x(x^4-2x^2-1)}{(x^4+1)^2}$$

$$x^4-2x^2-1=0$$
$$(x^2-1)^2=2$$
$$x^2-1=\pm\sqrt{2}$$
$$x^2=1\pm\sqrt{2}$$
$$x_{1,2}=\pm\sqrt{1+\sqrt{2}}\approx 1.5$$

$$f'(x)=0 \implies x_1=0,\quad x_{2,3}=\pm\sqrt{1+\sqrt{2}}\approx\pm 1.5,$$

$$f(0)=-1,\quad f\left(\pm\sqrt{1+\sqrt{2}}\right)=\frac{1}{2}(\sqrt{2}-1)\approx 0.2$$

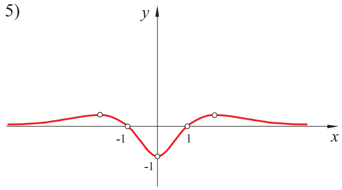
x	$\langle -\infty,-\sqrt{1+\sqrt{2}}\rangle$	$\langle -\sqrt{1+\sqrt{2}},0\rangle$	$\langle 0,\sqrt{1+\sqrt{2}}\rangle$	$\langle \sqrt{1+\sqrt{2}},+\infty\rangle$
$f'(x)$	$+$	$-$	$+$	$-$

$$f''(x)=\frac{[-2(x^4-2x^2-1)-2x(4x^3-4x)](x^4+1)^2+2x(x^4-2x^2-1)2(x^4+1)4x^3}{(x^4+1)^4}$$

$$f''(0)>0 \implies m(0,-1),\quad f''\left(\pm\sqrt{1+\sqrt{2}}\right)<0 \implies M\left(\pm\sqrt{1+\sqrt{2}},\frac{1}{2}(\sqrt{2}-1)\right)$$

infleksiju teško naći.

5)



6) $f(x) = \dfrac{1-4x^2}{4x^2(1+x^2)}$,

$D_f = \mathbf{R}^*$

$\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^+} f(x) = +\infty, \quad x = 0$ vertikalna asimptota

$f(x) = 0 \implies x_{1,2} = \pm\dfrac{1}{2}$ nultočka

$\lim\limits_{x\to\pm\infty} f(x) = 0 \implies y = 0$ horizontalna asimptota

x	$\left\langle -\infty, -\frac{1}{2}\right\rangle$	$\left\langle -\frac{1}{2}, 0\right\rangle$	$\left\langle 0, \frac{1}{2}\right\rangle$	$\left\langle \frac{1}{2}, +\infty\right\rangle$
$f(x)$	$-$	$+$	$+$	$-$

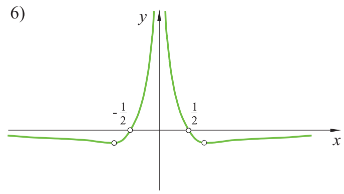
$$\begin{aligned}
f'(x) &= \frac{-8x(1+x^2)4x^2 - (1-4x^2)[8x(1+x^2)+4x^2\cdot 2x]}{16x^4(1+x^2)^2}\\
&= \frac{-32x^3(1+x^2) - (1-4x^2)8x(1+x^2+x^2)}{16x^4(1+x^2)}\\
&= \frac{-8x[4x^2(1+x^2)+(1-4x^2)(1+2x^2)]}{16x^4(1+x^2)}\\
&= \frac{-1-2x^2+4x^2+8x^4-4x^2-4x^4}{2x^3(1+x^2)^2}
\end{aligned}$$

$$f'(x) = \frac{4x^4-2x^2-1}{2x^3(1+x^2)}$$

$$4x^4-2x^2-1 = 0 \implies x^2 = \frac{2+\sqrt{4+16}}{8} = \frac{2+2\sqrt{5}}{8} = \frac{1+\sqrt{5}}{4}$$

$$\implies x = \pm\sqrt{\frac{1+\sqrt{5}}{4}} \approx \pm 0.9, \quad f(x) \approx -0.25$$

x	$\left\langle -\infty, -\sqrt{\frac{1+\sqrt{5}}{4}}\right\rangle$	$\left\langle -\sqrt{\frac{1+\sqrt{5}}{4}}, 0\right\rangle$	$\left\langle 0, \sqrt{\frac{1+\sqrt{5}}{4}}\right\rangle$	$\left\langle \sqrt{\frac{1+\sqrt{5}}{4}}, +\infty\right\rangle$
$f'(x)$	$-$	$+$	$-$	$+$



7) $f(x)=\dfrac{1}{x+1}+\dfrac{1}{x-1}=\dfrac{x+1+x-1}{(x+1)(x-1)}=\dfrac{2x}{x^2-1}$,

$D_f=\mathbf{R}\setminus\{-1,1\}$

$\lim\limits_{x\to-1^-}f(x)=-\infty,\quad \lim\limits_{x\to-1^+}f(x)=+\infty\quad x=-1$ vertikalna asimptota

$\lim\limits_{x\to1^-}f(x)=-\infty,\quad \lim\limits_{x\to1^+}f(x)=+\infty\quad x=1$ vertikalna asimptota

$\lim\limits_{x\to\pm\infty}f(x)=0\implies y=0$ horizontalna asimptota

$f(x)=0\implies x=0$ nultočka

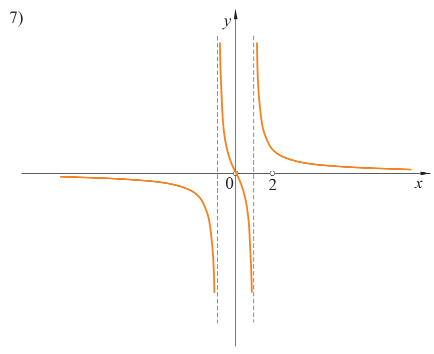
x	$\langle-\infty,-1\rangle$	$\langle-1,0\rangle$	$\langle0,1\rangle$	$\langle1,+\infty\rangle$
$f(x)$	$-$	$+$	$-$	$+$

$$f'(x)=\frac{2(x^2-1)-2x\cdot 2x}{(x^2-1)^2}=\frac{2x^2-2-4x^2}{(x^2-1)^2}=\frac{-2(x^2+1)}{(x^2-1)^2}=<0,\quad \forall x\in D_f$$

$$f''(x)=-2\frac{2x(x^2-1)^2-(x^2+1)2(x^2-1)2x}{(x^2-1)^4}=-2\frac{2x(x^2-1)(x^2-1-2x^2-2)}{(x^2-1)^4}$$

$$f''(x)=\frac{-4x(-x^2-3)}{(x^2-1)^3}=\frac{4x(x^2+3)}{(x^2-1)^3},\quad f''(x)=0\implies x=0$$

x	$\langle-\infty,-1\rangle$	$\langle-1,0\rangle$	$\langle0,1\rangle$	$\langle1,+\infty\rangle$
$f''(x)$	$-$	$+$	$-$	$+$



8) $f(x)=\dfrac{1-x}{x-2}$, (oblik $f(x)=\dfrac{ax+b}{cx+d}$) – razlomljena linearna funkcija

$D_f=\mathbf{R}\setminus\{2\}$

$$\lim_{x\to 2^-} f(x) = +\infty, \quad \lim_{x\to 2^+} f(x) = -\infty \implies x = 2 \quad \text{vertikalna asimptota}$$

$$f(x) = 0 \implies x = 1 \quad \text{nultočka}$$

$$\lim_{x\to\pm\infty} f(x) = -1 \implies y = -1 \quad \text{horizontalna asimptota}$$

x	$\langle -\infty, 1\rangle$	$\langle 1, 2\rangle$	$\langle 2, +\infty\rangle$
$f(x)$	$-$	$+$	$-$

$$f'(x) = \frac{-(x-2)-(1-x)}{(x-2)^2} = \frac{-x+2-1+x}{(x-2)^2} = \frac{1}{(x-2)^2} > 0, \quad \forall x \in D_f$$

$$f''(x) = \frac{-2}{(x-2)^3} = \frac{2}{(2-x)^3}$$

$$f(0) = -\frac{1}{2}$$

x	$\langle -\infty, 2\rangle$	$\langle 2, +\infty\rangle$
$f''(x)$	$+$	$-$

8)