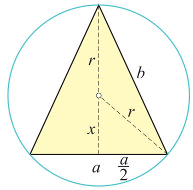
Zadatak 37. U kružnicu polumjera r upisan je jednakokračan trokut maksimalne površine. Kolike su mu stranice?

Rješenje.



Iz slike slijedi $x + r = h$ pa je:

$$\begin{aligned} x^2 + \frac{a^2}{4} &= r^2 \\ h^2 + \frac{a^2}{4} &= b^2 \\ \hline x^2 - h^2 &= r^2 - b^2 \\ (h-r)^2 - h^2 &= r^2 - b^2 \\ h^2 - 2hr + r^2 - h^2 &= r^2 - b^2 \\ 2hr &= b^2 \\ h &= \frac{b^2}{2r}. \end{aligned}$$

$\dfrac{a^2}{4} = b^2 - h^2 = b^2 - \dfrac{b^4}{4r^2} = \dfrac{b^2(4r^2 - b^2)}{4r^2} \implies a^2 = \dfrac{b^2(4r^2 - b^2)}{r^2} \implies$ $a = \dfrac{b}{r}\sqrt{4r^2 - b^2}$. Površina trokuta jednaka je $P = \dfrac{a \cdot h}{2} = \dfrac{b\sqrt{4r^2 - b^2} \cdot b^2}{4r^2} =$ $\dfrac{1}{4r^2} \cdot b^3\sqrt{4r^2 - b^2}$. Deriviramo izraz za površinu:

$$\frac{dP}{db} = \frac{1}{4r^2}\left(3b^2\sqrt{4r^2 - b^2} + b^3 \cdot \frac{-2b}{2\sqrt{4r^2 - b^2}}\right)$$

$= \dfrac{1}{4r^2}\left(\dfrac{6b^2(4r^2 - b^2) - 2b^4}{2\sqrt{4r^2 - b^2}}\right) = \dfrac{1}{8r^2} \cdot \dfrac{24r^2b^2 - 6b^4 - 2b^4}{\sqrt{4r^2 - b^2}}$. Izjednačimo to s nulom i dobijemo $3r^2b^2 - b^4 = 0 \implies b^2(3r^2 - b^2) = 0 \implies b^2 = 3r^2$.

Stranica a jednaka je $a^2 = \dfrac{b^2(4r^2 - b^2)}{r^2} = \dfrac{3r^2(4r^2 - 3r^2)}{r^2} = 3r^2$. Slijedi $a = b$, odnosno trokut je jednakostraničan.