Zadatak 21. Odredi područje definicije realne funkcije $f^n(x)$ ako je:

1) $f(x)=\dfrac{1}{1-x}$; **2)** $f(x)=\dfrac{1}{x+1}$.

Rješenje. **1)** $(f\circ f)(x)=f^2(x)=\dfrac{1}{1-\frac{1}{1-x}}=\dfrac{1-x}{1-x-1}=\dfrac{x-1}{x}$

$(f\circ f\circ f)(x)=f^3(x)=\dfrac{1}{1-\frac{x-1}{x}}=\dfrac{x}{x-x+1}=x$

$f^4(x)=\dfrac{1}{1-x}$

$$f^n(x)=\begin{cases} f(x), & n=3k-2,\\ \dfrac{x-1}{x}, & n=3k-1,\\ x, & n=3k,\ k\in\mathbf{N},\end{cases}$$

$$D(f^n)=\begin{cases} \mathbf{R}\setminus\{1\}, & n=3k-2,\\ \mathbf{R}\setminus\{0\}, & n=3k-1,\\ \mathbf{R}, & n=3k,\ k\in\mathbf{N}.\end{cases}$$

2) Imamo redom:

$f^2(x)=\dfrac{1}{\frac{1}{x+1}+1}=\dfrac{x+1}{x+2}$,

$f^3(x)=\dfrac{1}{\frac{x+1}{x+2}+1}=\dfrac{x+2}{2x+3}$,

$f^4(x)=\dfrac{1}{\frac{x+2}{2x+3}+1}=\dfrac{2x+3}{3x+5}$.

Postavljamo hipotezu da je $f^n(x)=\dfrac{a_nx+a_{n+1}}{a_{n+1}x+a_{n+2}}$, gdje je $a_{n+2}=a_n+a_{n+1}$, $a_1=0$, $a_2=1$.
Hipotezu dokazujemo matematičkom indukcijom. Bazu i već imamo.
Dokažimo još za $n+1$:

$$f^{n+1}(x)=\dfrac{1}{\frac{a_nx+a_{n+1}}{a_{n+1}x+a_{n+2}}+1}=\dfrac{a_{n+1}x+a_{n+2}}{a_nx+a_{n+1}+a_{n+1}x+a_{n+2}}$$

$$=\dfrac{a_{n+1}x+a_{n+2}}{(a_n++a_{n+1})x+(a_{n+1}+a_{n+2})}=\dfrac{a_{n+1}x+a_{n+2}}{a_{n+2}x+a_{n+3}}.$$

$D(f^n)=\mathbf{R}\setminus\{b_n\}$, $b_n=-\dfrac{a_{n+2}}{a_{n+1}}$.