

Zadatak 3.

Metodom parcijalne integracije odredi:

- 1) $\int x \cos^2 x dx$; 2) $\int \frac{x}{\sin^2 x} dx$;
- 3) $\int \frac{x dx}{\cos^2 x}$; 4) $\int \sin^3 x dx$;
- 5) $\int x \cos 3x dx$; 6) $\int x^3 \sin x dx$;
- 7) $\int \frac{\ln x}{x^3} dx$; 8) $\int \frac{\ln x}{\sqrt{x}} dx$;
- 9) $\int x e^x \sin x dx$.

Rješenje.

1) $\int x \cos^2 x dx = \int x \left(\frac{1 + \cos 2x}{2} \right) dx = \int \frac{x}{2} dx + \int \frac{x}{2} \cos 2x dx = \frac{x^2}{4} + \frac{1}{2} \int x \cos 2x dx = \begin{cases} x=u & \cos 2x dx = dv \\ dx=du & \frac{1}{2} \sin 2x = v \end{cases} = \frac{x^2}{4} + \frac{1}{2} \left(\frac{x}{2} \sin 2x - \frac{1}{2} \int \sin 2x dx \right) = \frac{x^2}{4} + \frac{x}{4} \sin 2x - \frac{1}{4} \left(-\frac{1}{2} \right) \cos 2x + C = \frac{x^2}{4} + \frac{x}{2} \sin 2x + \frac{1}{8} \cos 2x + C$;

2) $\int \frac{x}{\sin^2 x} dx = \begin{cases} x=u & \frac{dx}{\sin^2 x} = dv \\ dx=du & -\operatorname{ctg} x = v \end{cases} = -x \operatorname{ctg} x + \int \operatorname{ctg} x dx = -x \operatorname{ctg} x + \int \frac{\cos x}{\sin x} dx = -x \operatorname{ctg} x + \int \frac{d(\sin x)}{\sin x} = -x \operatorname{ctg} x + \ln |\sin x| + C$;

3) $\int \frac{x dx}{\cos^2 x} = \begin{cases} x=u & \frac{dx}{\cos^2 x} = dv \\ dx=du & \operatorname{tg} x = v \end{cases} = x \operatorname{tg} x - \int \operatorname{tg} x dx = x \operatorname{tg} x - \int \frac{\sin x}{\cos x} dx = x \operatorname{tg} x + \int \frac{d(\cos x)}{\cos x} = x \operatorname{tg} x + \ln |\cos x| + C$;

4) $\int \sin^3 x dx = \int \sin^2 x \sin x dx = - \int (1 - \cos^2 x) d(\cos x) = \begin{cases} \cos x = t & \\ d(\cos x) = dt & \end{cases} = \int (t^2 - 1) dt = \frac{1}{3} t^3 - t + C = \frac{1}{3} \cos^3 x - \cos x + C = \cos x \left(\frac{1}{3} \cos^2 x - 1 \right) + C = \cos x \left(\frac{1}{3} \cos^2 x - \sin^2 x - \cos^2 x \right) + C = \cos x \left(-\frac{2}{3} \cos^2 x - \sin^2 x \right) + C = -\frac{2}{3} \cos^3 x - \sin^2 x \cos x + C$;

ili $\int \sin^3 x dx = \int \sin^2 x \sin x dx = \begin{cases} \sin^2 x = u & \sin x dx = dv \\ 2 \sin x \cos x dx = du & -\cos x = v \end{cases} = -\sin^2 x \cos x + \int 2 \sin x \cos^2 x dx = -\sin^2 x \cos x - 2 \int \cos^2 x d(\cos x) = -\sin^2 x \cos x - \frac{2}{3} \cos^3 x + C$;

5) $\int x \cos 3x dx = \begin{cases} x=u & \cos 3x dx = dv \\ dx=du & \frac{1}{3} \sin 3x = v \end{cases} = \frac{1}{3} x \sin 3x - \frac{1}{3} \int \sin 3x dx = \frac{1}{3} x \sin 3x + \frac{1}{9} \cos 3x + C$;

$$\begin{aligned}
 6) \int x^3 \sin x \, dx &= \left\{ \begin{array}{l} x^3 = u \\ 3x^2 \, dx = du \end{array} \quad \begin{array}{l} \sin x \, dx = dv \\ -\cos x = v \end{array} \right\} = -x^3 \cos x + 3 \int x^2 \cos x \, dx = \\
 &\left\{ \begin{array}{l} x^2 = u \\ 2x \, dx = du \end{array} \quad \begin{array}{l} \cos x \, dx = dv \\ \sin x = v \end{array} \right\} = -x^3 \cos x + 3x^2 \sin x - 3 \cdot 2 \int x \sin x \, dx \\
 &= \left\{ \begin{array}{l} x = u \\ dx = du \end{array} \quad \begin{array}{l} \sin x \, dx = dv \\ -\cos x = v \end{array} \right\} = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \int \cos x \, dx = \\
 &-x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C = (3x^2 - 6) \sin x - (x^3 - 6x) \cos x + C;
 \end{aligned}$$

$$\begin{aligned}
 7) \int \frac{\ln x}{x^3} \, dx &= \left\{ \begin{array}{l} \ln x = u \\ \frac{dx}{x} = du \end{array} \quad \begin{array}{l} \frac{dx}{x^3} = dv \\ -\frac{1}{2x^2} = v \end{array} \right\} = -\frac{1}{2x^2} \ln |x| + \frac{1}{2} \int \frac{dx}{x^3} = \\
 &-\frac{1}{2x^2} \ln |x| + \frac{1}{2} \left(-\frac{1}{2} \right) \frac{1}{x^2} + C = -\frac{1}{2x^2} \ln |x| - \frac{1}{4x^2} + C;
 \end{aligned}$$

$$\begin{aligned}
 8) \int \frac{\ln x}{\sqrt{x}} \, dx &= \left\{ \begin{array}{l} \ln x = u \\ \frac{dx}{x} = du \end{array} \quad \begin{array}{l} x^{-\frac{1}{2}} dx = dv \\ v = 2\sqrt{x} \end{array} \right\} = 2\sqrt{x} \ln x - 2 \int \frac{dx}{\sqrt{x}} = \\
 &2\sqrt{x} \ln x - 4\sqrt{x} + C;
 \end{aligned}$$

$$\begin{aligned}
 9) \int x e^x \sin x \, dx &= \left\{ \begin{array}{l} xe^x = u \\ (e^x + xe^x) \, dx = du \end{array} \quad \begin{array}{l} \sin x \, dx = dv \\ -\cos x = v \end{array} \right\} = -xe^x \cos x + \\
 &\int (x+1)e^x \cos x \, dx = \left\{ \begin{array}{l} (x+1)e^x = u \\ (e^x + (x+1)e^x) \, dx = du \end{array} \quad \begin{array}{l} \cos x \, dx = dv \\ \sin x = v \end{array} \right\} = -xe^x \cos x + \\
 &(x+1)e^x \sin x - \int (x+2)e^x \sin x \, dx = -xe^x \cos x + (x+1)e^x \sin x - \int xe^x \sin x \, dx = \\
 &2 \int e^x \sin x \, dx \implies xe^x \sin x \, dx = -\frac{1}{2}xe^x \cos x + \frac{1}{2}(x+1)e^x \sin x - \int e^x \sin x \, dx \\
 &\int e^x \sin x \, dx = \left\{ \begin{array}{l} \sin x = u \\ \cos x \, dx = du \end{array} \quad \begin{array}{l} e^x \, dx = dv \\ e^x = v \end{array} \right\} = e^x \sin x - \int e^x \cos x \, dx \\
 &= \left\{ \begin{array}{l} \cos x = u \\ -\sin x \, dx = dv \end{array} \quad \begin{array}{l} e^x \, dx = dv \\ e^x = v \end{array} \right\} = e^x \cos x - (e^x \cos x + \int e^x \sin x \, dx) = \\
 &e^x \sin x - e^x \cos x - \int e^x \sin x \, dx \implies \int e^x \sin x \, dx = \frac{1}{2}e^x(\sin x - \cos x) \implies \\
 &\int xe^x \sin x \, dx = -\frac{1}{2}xe^x \cos x + \frac{1}{2}(x+1)e^x \sin x - \frac{1}{2}e^x(\sin x - \cos x) + C = \\
 &\frac{1}{2}e^x(-x \cos x + x \sin x + \sin x - \sin x + \cos x) + C = \frac{1}{2}e^x(\cos x + x \sin x - x \cos x) + C.
 \end{aligned}$$