Zadatak 2. Izračunaj volumen rotacijskog tijela što nastaje rotacijom oko x-osi lika omeđenog krivuljama:

1) $y = x^2$, $y = -x^2 + 4x$;

2) $y^2 = |x|$, $y = 0$, $x = -1$ i $x = 4$;

3) $y = \frac{1}{x}$, $x = 2$, $y = 0$ i $2x - 2y + 3 = 0$;

4) $y = \sin \pi x$ i $2x + y - 2 = 0$;

5) $y = x^2$, $y = \frac{1}{x}$, $y = 0$ i $x = e$;

6) $y = \sqrt{x}$, $y = \sqrt{2 - x}$ i $y = 0$;

7) $y = \sqrt{2x + 1}$, $y = 0$ i $3x + 2y - 18 = 0$.

Rješenje. **1)** $x^2 = -x^2 + 4x \implies 2x^2 - 4x = 0 \implies x^2 - 2x = 0 \implies x(x-2) = 0 \implies x_1 = 0, \quad x_2 = 2$.

$V = V_1 - V_2$.

$V_1 = \pi \int_0^2 (-x^2 + 4x)^2 dx = \pi \int_0^2 (x^4 - 8x^3 + 16x^2) dx = \pi\Big(\frac{1}{5}x^5 - 2x^4 + \frac{16}{3}x^3\Big)\Big|_0^2 = \pi\Big(\frac{32}{5} - 32 + \frac{128}{3}\Big) = \frac{96 - 480 + 640}{15}\pi = \frac{256}{15}\pi$.

$V_2 = \pi \int_0^2 (x^2)^2 dx = \pi \int_0^2 x^4 dx = \pi\Big(\frac{x^5}{5}\Big)\Big|_0^2 = \frac{32}{5}\pi = \frac{96}{15}\pi$.

$V = V_1 - V_2 = \frac{256\pi}{15} - \frac{96\pi}{15} = \frac{160}{15}\pi = \frac{32}{3}\pi$.

2) $V = V_1 + V_2$.

$V_1 = \pi \int_{-1}^0 (\sqrt{-x})^2 dx = \pi \int_{-1}^0 (-x) dx = \pi\Big(-\frac{x^2}{2}\Big)\Big|_{-1}^0 = \frac{\pi}{2}$.

$V_2 = \pi \int_0^4 (\sqrt{x})^2 dx = \pi \int_0^4 x dx = \pi\Big(\frac{x^2}{2}\Big)\Big|_0^4 = 8\pi$.

$V = V_1 + V_2 = 8\pi + \frac{\pi}{2} = \frac{17}{2}\pi$.

3) $2x - 2y + 3 = 0 \implies 2y = 2x + 3 \implies y = x + \frac{3}{2}$.

$\frac{1}{x} = x + \frac{3}{2} \implies 2x^2 + 3x - 2 = 0 \implies x_1 = \frac{1}{2}, \quad x_2 = 2$.

$V_1 = \pi \int_{1/2}^2 \Big(\frac{1}{x}\Big)^2 dx = \pi \int_{1/2}^2 \frac{dx}{x^2} = \pi\Big(-\frac{1}{x}\Big)\Big|_{1/2}^2 = \pi\Big(-\frac{1}{2} + 2\Big) = \frac{3}{2}\pi$.

$V_2 = \pi \int_{-3/2}^{1/2} \Big(x + \frac{3}{2}\Big)^2 dx = \pi \int_{-3/2}^{1/2} \Big(x^2 + 3x + \frac{9}{4}\Big) dx = \pi\Big(\frac{1}{3}x^3 + \frac{3}{2}x^2 + \frac{9}{4}x\Big)\Big|_{-3/2}^{1/2} = \pi\Big(\frac{1}{24} + \frac{3}{8} + \frac{9}{8} + \frac{9}{8} - \frac{27}{8} + \frac{27}{8}\Big) = \pi\Big(\frac{1}{24} - \frac{15}{8} + \frac{9}{2}\Big) = \pi\Big(\frac{1}{24} - \frac{45}{24} + \frac{9}{2}\Big) = \pi\Big(\frac{9}{2} - \frac{11}{6}\Big) = \frac{16}{6}\pi = \frac{8}{3}\pi$.

$V = V_1 + V_2 = \frac{3}{2}\pi + \frac{8}{3}\pi = \frac{9 + 16}{6}\pi = \frac{25}{6}\pi$.

4) $y = \sin \pi x$, $2x + y - 2 = 0$, $y = -2x + 2$.

Za $x_1 = \frac{1}{2}$ je $\sin \pi x = 1$ i $-2x + 2 = 1$.
Za $x_2 = 1$ je $\sin \pi x = 0$ i $-2x + 2 = 0$.
$V = 2(V_1 - V_2)$.
$V_1 = \pi \int_{1/2}^{1} (\sin \pi x)^2 dx = \pi \int_{1/2}^{1} \sin^2 \pi x dx = \frac{\pi}{2} \int_{1/2}^{1} (1 - \cos 2\pi x) dx =$
$\frac{\pi}{2}(x - \sin 2\pi x)\Big|_{1/2}^{1} = \frac{\pi}{2}\left(1 - \frac{1}{2}\right) = \frac{\pi}{4}$.
$V_2 = \pi \int_{1/2}^{1} (-2x + 2)^2 dx = \pi \int_{1/2}^{1} (4x^2 - 8x + 4) dx = 4\pi \int_{1/2}^{1} (x^2 -$
$2x + 1) dx = 4\pi\left(\frac{1}{3}x^3 - x^2 + x\right)\Big|_{1/2}^{1} = 4\pi\left(\frac{1}{3} - 1 + 1 - \frac{1}{24} + \frac{1}{4} - \frac{1}{2}\right) =$
$4\pi \cdot \frac{8 - 1 + 6 - 12}{24} = 4\pi \cdot \frac{1}{24} = \frac{\pi}{6}$.
$V = 2(V_1 - V_2) = 2\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = 2 \cdot \frac{3\pi - 2\pi}{12} = \frac{\pi}{6}$.

5) $V = V_1 + V_2$.
$V_1 = \pi \int_0^1 (x^2)^2 dx = \pi \int_0^1 x^4 dx = \frac{\pi}{5}(x^5)\Big|_0^1 = \frac{\pi}{5}$.
$V_2 = \pi \int_1^e \left(\frac{1}{x}\right)^2 dx = \pi \int_1^e \frac{dx}{x^2} = \pi\left(-\frac{1}{x}\right)\Big|_1^e = \left(1 - \frac{1}{e}\right)\pi$.
$V = V_1 + V_2 = \pi - \frac{1}{e}\pi + \frac{1}{5}\pi = \left(\frac{6}{5} - \frac{1}{e}\right)\pi$.

6) $V = V_1 + V_2$.
$V_1 = \pi \int_0^1 (\sqrt{x})^2 dx = \pi \int_0^1 x dx = \pi \cdot \left(\frac{x^2}{2}\right)\Big|_0^1 = \frac{\pi}{2}$.
$V_2 = \pi\Big|_1^2 (\sqrt{2 - x})^2 dx = \pi \int_1^2 (2-x) dx = \pi\left(2x - \frac{x^2}{2}\right)\Big|_1^2 = \pi\left(4 - 2 - 2 + \frac{1}{2}\right) =$
$\frac{\pi}{2}$.
$V = \pi$.

7) $\sqrt{2x + 1} = 9 - \frac{3}{2}x \implies x = 4$.
$V = V_1 + V_2$.
$V_1 = \pi \int_{-1/2}^{4} (\sqrt{2x + 1})^2 dx = \pi \int_{-1/2}^{4} (2x + 1) dx = \pi(x^2 + x)\Big|_{-1/2}^{4} =$
$\pi\left(16 + 4 - \frac{1}{4} + \frac{1}{2}\right) = \frac{81}{4}\pi$.
$V_2 = \pi \int_4^6 \left(-\frac{3}{2}x + 9\right)^2 dx = \pi \int_4^6 \left(\frac{18 - 3x}{2}\right)^2 dx = \frac{9}{4}\pi \int_4^6 (6 - x)^2 dx =$
$\frac{9}{4}\pi \int_4^6 (36 - 12x + x^2) dx = \frac{9}{4}\pi\left(36x - 6x^2 + \frac{x^3}{3}\right)\Big|_4^6 = \frac{9}{4}\pi\Big[36 \cdot 6 - 36 \cdot 6 +$
$\frac{36 \cdot 6}{3} - 36 \cdot 4 + 6 \cdot 16 - \frac{64}{3}\Big] = \frac{9}{4}\pi\left(\frac{19 \cdot 8}{3} - 48\right) = (3 \cdot 19 \cdot 2 - 9 \cdot 12)\pi = 6\pi$.
$V = V_1 + V_2 = \frac{81}{4}\pi + 6\pi = \frac{105}{4}\pi$.