Zadatak 3. Izračunaj volumen rotacijskog tijela što nastaje rotacijom oko y-osi lika omeđenog krivuljama:

1) $y = x^2$, $y = 0$, $x = 0$ i $x = 1$;

2) $y^2 = 4x$ i $x = 1$;

3) $y = \frac{1}{2}x^2 + 2x + 2$ i $y = 2$.

Rješenje.

1) $y = x^2 \implies x = \sqrt{y}$.

$$V = \pi \int_0^1 1^2 dy - \pi \int_0^1 (\sqrt{y})^2 dy = \pi \int_0^1 dy - \pi \int_0^1 y dy = \pi \left[(y) \Big|_0^1 - \left(\frac{y^2}{2} \right) \Big|_0^1 \right] = \frac{\pi}{2}.$$

2) $y^2 = 4x \implies x = \frac{y^2}{4}$.

$$V = \pi \int_{-2}^2 1^2 dy - \pi \int_{-2}^2 \left(\frac{y^2}{4} \right)^2 dy = 2\pi \int_0^2 dy - 2\pi \int_0^2 \left(\frac{y^2}{4} \right)^2 dy = 2\pi \int_0^2 dy - \frac{\pi}{8} \int_0^2 y^4 dy = 2\pi (y) \Big|_0^2 - \frac{\pi}{8} \frac{1}{5} (y^5) \Big|_0^2 = 4\pi - \frac{\pi}{5} \frac{1}{8} \cdot 32 = 4\pi - \frac{4\pi}{5} = \frac{20\pi - 4\pi}{5} = \frac{16\pi}{5}.$$

3) $y = \frac{1}{2}x^2 + 2x + 2 \implies y = \frac{1}{2}(x^2 + 4x + 4) = \frac{1}{2}(x+2)^2 \implies (x+2)^2 = 2y \implies x + 2 = \pm\sqrt{2y} \implies x = \pm\sqrt{2y} - 2$.

$$V = \pi \int_0^2 (\sqrt{2y} + 2)^2 dy - \pi \int_0^2 (2 - \sqrt{2y})^2 dy = \pi \int_0^2 (2y + 4\sqrt{2y} + 4) dy - \pi \int_0^2 (4 - 4\sqrt{2y} + 2y) dy = \pi \int_0^2 (2y + 4\sqrt{2y} + 4 - 4 + 4\sqrt{2y} - 2y) dy = 8\pi\sqrt{2} \int_0^2 \sqrt{y} dy = 8\pi\sqrt{2} \int_0^2 y^{\frac{1}{2}} dy = 8\pi\sqrt{2} \frac{2}{3} \left(y^{\frac{3}{2}} \right) \Big|_0^2 = \frac{16\pi}{3}\sqrt{2} \cdot 2\sqrt{2} = \frac{64\pi}{3}.$$