Zadatak 10. Nađi primitivnu funkciju svake od navedenih funkcija uz dani uvjet:

1) $f(x) = \sin\frac{x}{2}\cos x\,,\ F(\pi) = 1\,;$

2) $f(x) = \sin x\cos\frac{x}{2}\,,\ F(-2\pi) = \frac{4}{3}\,;$

3) $f(x) = \dfrac{1}{\cos^2(2x - \frac{\pi}{4})}\,,\ F(\frac{\pi}{4}) = 1\,;$

4) $f(x) = \dfrac{1}{x-3} + e^{2x+1}\,,\ F(2) = e^5\,;$

5) $f(x) = \dfrac{1}{x} + 2^{3x}\,,\ F(1) = 3\,;$

6) $f(x) = x^2 + \frac{1}{2}\operatorname{ctg}\frac{x}{2}\,,\ F(\pi) = \pi^3\,;$

7) $f(x) = e^{\sin x}\cdot\cos x\,,\ F(\frac{\pi}{2}) = e + 3\,;$

8) $f(x) = (x - \frac{1}{2})e^{x^2-x}\,,\ F(1) = -\frac{1}{2}\,.$

Rješenje. **1)** $f(x) = \sin\frac{x}{2}\cos x\,,\ F(\pi) = 1\,;$

$$\begin{aligned}
F(x) &= \int \sin\frac{x}{2}\cos x dx = \int \sin\frac{x}{2}\left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right)dx \\
&= \int \sin\frac{x}{2}\left(2\cos^2\frac{x}{2} - 1\right)dx = 2\int\left(1 - 2\cos^2\frac{x}{2}\right)d\left(\cos\frac{x}{2}\right) \\
&= 2\cos\frac{x}{2} - \frac{4}{3}\cos^3\frac{x}{2} + C \\
F(\pi) &= 2\cos\frac{\pi}{2} - \frac{4}{3}\cos^3\frac{\pi}{2} + C = C = 1 \\
&\Longrightarrow F(x) = 2\cos\frac{x}{2} - \frac{4}{3}\cos^3\frac{x}{2} + 1;
\end{aligned}$$

2) $f(x) = \sin x\cos\frac{x}{2}\,,\ F(-2\pi) = \frac{4}{3}\,;$

$$\begin{aligned}
F(x) &= \int \sin x\cos\frac{x}{2}dx = \int \frac{1}{2}\left[\sin\frac{x}{2} + \sin\frac{3x}{2}\right]dx \\
&= \frac{1}{2}\int \sin\frac{x}{2}dx + \frac{1}{2}\int \sin\frac{3x}{2}dx = -\cos\frac{x}{2} - \frac{1}{3}\cos\frac{3x}{2} + C
\end{aligned}$$

$$F(-2\pi) = -\cos(-\pi) - \frac{1}{3}\cos(-3\pi) + C = 1 + \frac{1}{3} + C = \frac{4}{3} \Longrightarrow C = 0$$

$$\Longrightarrow F(x) = -\cos\frac{x}{2} - \frac{1}{3}\cos\frac{3x}{2};$$

3) $f(x) = \dfrac{1}{\cos^2(2x - \frac{\pi}{4})}$, $F(\frac{\pi}{4}) = 1$;

$$F(x) = \int \frac{dx}{\cos^2\left(2x - \frac{\pi}{4}\right)} = \frac{1}{2}\int \frac{d\left(2x - \frac{\pi}{4}\right)}{\cos^2\left(2x - \frac{\pi}{4}\right)} = \frac{1}{2}\operatorname{tg}\left(2x - \frac{\pi}{4}\right) + C$$

$$F\left(\frac{\pi}{4}\right) = \frac{1}{2}\operatorname{tg}\frac{\pi}{4} + C = \frac{1}{2} + C = 1 \implies C = \frac{1}{2}$$

$$\implies F(x) = \frac{1}{2}\operatorname{tg}\left(2x - \frac{\pi}{4}\right) + \frac{1}{2};$$

4) $f(x) = \dfrac{1}{x-3} + e^{2x+1}$, $F(2) = e^5$;

$$F(x) = \int \left(\frac{1}{x-3} + e^{2x+1}\right)dx = \ln|x-3| + \frac{1}{2}e^{2x+1} + C$$

$$F(2) = \frac{1}{2}e^5 + C = e^5 \implies C = \frac{1}{2}e^5$$

$$\implies F(x) = \ln|x-3| + \frac{1}{2}e^{2x+1} + \frac{1}{2}e^5;$$

5) $f(x) = \dfrac{1}{x} + 2^{3x}$, $F(1) = 3$;

$$F(x) = \int \left(\frac{1}{x} + 2^{3x}\right)dx = \ln|x| + \frac{1}{3\ln 2}\cdot 2^{3x} + C$$

$$F(1) = \frac{8}{3\ln 2} + C = 3 \implies C = 3 - \frac{8}{3\ln 2}$$

$$\implies F(x) = \ln|x| + \frac{2^{3x}}{3\ln 2} - \frac{8}{3\ln 2} + 3;$$

6) $f(x) = x^2 + \frac{1}{2}\operatorname{ctg}\frac{x}{2}$, $F(\pi) = \pi^3$;

$$F(x) = \int \left(x^2 + \frac{1}{2}\operatorname{ctg}\frac{x}{2}\right)dx = \frac{x^3}{3} + \frac{1}{2}\int \frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}dx$$

$$= \frac{x^3}{3} + \int \frac{d(\sin\frac{x}{2})}{\sin\frac{x}{2}} = \frac{x^3}{3} + \ln\left|\sin\frac{x}{2}\right| + C$$

$$F(\pi) = \frac{\pi^3}{3} + C = \pi^3 \implies C = \frac{2\pi^3}{3}$$

$$\implies F(x) = \frac{x^3}{3} + \ln\left|\sin\frac{x}{2}\right| + \frac{2}{3}\pi^3;$$

7) $f(x) = e^{\sin x}\cdot\cos x$, $F(\frac{\pi}{2}) = e + 3$;

$$F(x) = \int e^{\sin x}\cos x\,dx = \int e^{\sin x}d(\sin x) = e^{\sin x} + C$$

$$F\left(\frac{\pi}{2}\right) = e + C = e + 3 \implies C = 3 \implies F(x) = e^{\sin x} + 3;$$

8) $f(x) = (x - \frac{1}{2})e^{x^2-x}$, $F(1) = -\frac{1}{2}$;

$$F(x) = \int \left(x - \frac{1}{2}\right)e^{x^2-x}dx = \frac{1}{2}\int (2x-1)e^{x^2-x}dx$$

$$= \frac{1}{2}\int e^{x^2-x}d(x^2-x) = \frac{1}{2}e^{x^2-x} + C$$

$$F(1) = \frac{1}{2} + C = -\frac{1}{2} \implies C = -1 \implies F(x) = \frac{1}{2}e^{x^2-x} - 1.$$