Zadatak 5. Računajući po definiciji dokaži:

1) $\lim\limits_{n\to\infty}\frac{3n+1}{2n-1}=\frac{3}{2}$; **2)** $\lim\limits_{n\to\infty}\frac{2n+1}{n+1}=2$;

3) $\lim\limits_{n\to\infty}\frac{3n-2}{2n}=1.5$; **4)** $\lim\limits_{n\to\infty}\frac{1}{n^2+1}=0$.

Rješenje. **1)** $\lim\limits_{n\to\infty}\frac{3n+1}{2n-1}=\frac{3}{2}$;

$$\left|\frac{3n+1}{2n-1}-\frac{3}{2}\right|=\left|\frac{6n+2-6n+3}{2(2n-1)}\right|=\frac{5}{2(2n-1)}<\varepsilon$$
$$\Longrightarrow\ \frac{2(2n-1)}{5}>\frac{1}{\varepsilon}\ \Longrightarrow\ 2n-1>\frac{5}{2\varepsilon}$$
$$\Longrightarrow\ 2n>\frac{5}{2\varepsilon}+1\ \Longrightarrow\ n>\frac{5}{4\varepsilon}+\frac{1}{2};$$

Dakle za svaki $\varepsilon>0$ postoji prirodan broj $n_0=\left[\frac{5}{4\varepsilon}+\frac{1}{2}\right]$ takav da za sve $n>n_0$ vrijedi $\left|\frac{3n+1}{2n-1}-\frac{3}{2}\right|<\varepsilon$.

2) $\lim\limits_{n\to\infty}\frac{2n+1}{n+1}=2$;

$$\left|\frac{2n+1}{n+1}-2\right|=\left|\frac{2n+1-2n-2}{n+1}\right|=\left|-\frac{1}{n+1}\right|=\frac{1}{n+1}<\varepsilon$$
$$\Longrightarrow\ n+1>\frac{1}{\varepsilon}\ \Longrightarrow\ n>\frac{1}{\varepsilon}-1;$$

Dakle za svaki $\varepsilon>0$ postoji prirodan broj $n_0=\left[\frac{1}{\varepsilon}-1\right]$ takav da za sve $n>n_0$ vrijedi $\left|\frac{2n+1}{n+1}-2\right|<\varepsilon$.

3) $\lim\limits_{n\to\infty}\frac{3n-2}{2n}=\frac{3}{2}$;

$$\left|\frac{3n-2}{2n}-\frac{3}{2}\right|=\left|\frac{3n-2-3n}{2n}\right|=\left|\frac{-2}{2n}\right|=\frac{1}{n}<\varepsilon\ \Longrightarrow\ n>\frac{1}{\varepsilon};$$

Dakle za svaki $\varepsilon>0$ postoji prirodan broj $n_0=\left[\frac{1}{\varepsilon}\right]$ takav da za sve $n>n_0$ vrijedi $\left|\frac{3n-2}{2n}-\frac{3}{2}\right|<\varepsilon$.

4) $\lim\limits_{n\to\infty}\frac{1}{n^2+1}=0$;

$$\left|\frac{1}{n^2+1}-0\right|=\frac{1}{n^2+1}<\varepsilon\ \Longrightarrow\ n^2+1>\varepsilon\ \Longrightarrow\ n^2>\varepsilon-1$$
$$\Longrightarrow\ n>\sqrt{|\varepsilon-1|};$$

Dakle za svaki $\varepsilon>0$ postoji prirodan broj $n_0=\left[\sqrt{|\varepsilon-1|}\right]$ takav da za sve $n>n_0$ vrijedi $\left|\frac{1}{n^2+1}-0\right|<\varepsilon$.