

Zadatak 19. Je li niz (a_n) omeđen:

$$1) a_n = \frac{n+1}{\log_2(n+2)};$$

$$2) a_n = n \cdot \operatorname{tg} \frac{(2n+1)\pi}{4};$$

$$3) a_n = 3^n + 2^{-n};$$

$$4) a_n = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{n(n+1)};$$

$$5) a_n = \frac{1+2+3+\dots+n}{n+1};$$

$$6) a_n = \frac{1^2+2^2+\dots+n^2}{n^2(n+2)}?$$

Rješenje.

$$1) a_n = \frac{n+1}{\log_2(n+2)} = \frac{\log_2 2^{n+1}}{\log_2(n+2)};$$

$$a_1 = \frac{2}{\log_2 3} \approx 1.262, a_2 = \frac{3}{\log_2 4} = 1.5, a_3 = \frac{4}{\log_2 5} \approx 1.723, \dots$$

Niz je monotono rastući,

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{\log_2 2^{n+1}}{\log_2(n+2)} = \infty$$

Niz nije omeđen.

$$2) a_1 = -1, a_2 = 2, a_3 = -3, a_4 = 4, a_5 = -5, \dots$$

Niz a_n ima dva podniza a_{2k-1} i a_{2k} , $k \in \mathbf{N}$, od koji prvi nije omeđen odozdo, a drugi odozgo, pa ni sam niz nije omeđen.

3) Niz monotono raste i

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} (3^n + 2^{-n}) = \infty$$

te nije omeđen.

$$4) a_n = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{n(n+1)} = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \dots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1};$$

$$a_1 = \frac{1}{2} \text{ i}$$

$$\lim_{n \rightarrow \infty} \left(1 - \frac{1}{n+1}\right) = 1;$$

$$\text{Dakle } \frac{1}{2} \leq a_n < 1.$$

$$5) a_n = \frac{1+2+3+\dots+n}{n+1} = \frac{\frac{n}{2}(1+n)}{n+1} = \frac{n}{2};$$

Niz monotono raste i

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left(\frac{n}{2}\right) = \infty$$

te nije omeđen.

$$\begin{aligned}
 \text{6) } a_n &= \frac{1^2 + 2^2 + \dots + n^2}{n^2(n+2)} = \frac{\frac{n(n+1)(2n+1)}{6}}{n^2(n+2)} = \frac{(n+1)(2n+1)}{6n(n+2)} = \\
 &= \frac{2n^2 + 3n + 1}{6n(n+2)} = \frac{2n^2 + 4n - n + 1}{6n(n+2)} = \frac{1}{3} - \underbrace{\frac{n-1}{6n(n+2)}}_{\geq 0} \leq \frac{1}{3};
 \end{aligned}$$

Niz nije monoton ali vrijedi da je izraz $\frac{n-1}{6n(n+2)}$ manji od $\frac{1}{3}$. Provjerimo:

$$\frac{n-1}{6n(n+2)} < \frac{1}{3} \iff 3n-3 < 6n^2+12n \iff 6n^2+9n+3 > 0;$$

te su svi članovi niza iz $\left\langle 0, \frac{1}{3} \right\rangle$ i niz je omeđen.