Zadatak 8. Koliko je

1) $(0.2)^{\log_{\sqrt{5}}(\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots)}$;

2) $\sqrt{3\sqrt{5\sqrt{3\sqrt{5\sqrt{\ldots}}}}}$;

3) $\sqrt{2\sqrt{2\sqrt{2\sqrt{\ldots}}}}$;

4) $\sqrt{3\sqrt{4\sqrt{3\sqrt{4\sqrt{\ldots}}}}}$?

Rješenje. **1)** $\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots=\frac{\frac{1}{4}}{1-\frac{1}{2}}=\frac{1}{2}$

$$(0.2)^{\log_{\sqrt{5}}(\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots)}=(5^{-1})^{2\log_5\frac{1}{2}}=5^{-2\log_5(2^{-1})}=5^{\log_5(2^2)}=4;$$

2)

$$x=\sqrt{3\sqrt{5\sqrt{3\sqrt{5\sqrt{\ldots}}}}}=3^{\frac{1}{2}}5^{\frac{1}{4}}3^{\frac{1}{8}}5^{\frac{1}{16}}\ldots=3^{\frac{1}{2}+\frac{1}{8}+\cdots}5^{\frac{1}{4}+\frac{1}{16}+\cdots},$$

Eksponenti su geometrijski redovi s kvocijentima manjim od 1, čije su sume:

$$\frac{1}{2}+\frac{1}{8}+\ldots=\frac{\frac{1}{2}}{1-\frac{1}{4}}=\frac{4}{6}=\frac{2}{3},$$

$$\frac{1}{4}+\frac{1}{16}+\ldots=\frac{\frac{1}{4}}{1-\frac{1}{4}}=\frac{4}{12}=\frac{1}{3};$$

te gornji izraz za x glasi:

$$x=3^{\frac{2}{3}}5^{\frac{1}{3}}=\sqrt[3]{3^2\cdot 5}=\sqrt[3]{45};$$

3) $\sqrt{2\sqrt{2\sqrt{2\sqrt{\ldots}}}}=2^{\frac{1}{2}}2^{\frac{1}{4}}2^{\frac{1}{8}}\ldots 2^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots}==2^{\frac{\frac{1}{2}}{1-\frac{1}{2}}}=2^1=2;$

4) $x=\sqrt{3\sqrt{4\sqrt{3\sqrt{4\sqrt{\ldots}}}}}=3^{\frac{1}{2}}4^{\frac{1}{4}}3^{\frac{1}{8}}4^{\frac{1}{16}}\ldots=3^{\frac{1}{2}+\frac{1}{8}+\cdots}4^{\frac{1}{4}+\frac{1}{16}+\cdots}$

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te gornji izraz za x glasi:

$$x=3^{\frac{2}{3}}\cdot 4^{\frac{1}{3}}=\sqrt[3]{3^2\cdot 4}=\sqrt[3]{36}.$$