

Zadatak 9. Riješi sljedeće jednadžbe:

- 1) $2^{x+x^2+x^3+\dots} = 2\sqrt{2};$
- 2) $x^{\log(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots)} = 1;$
- 3) $1 + \log_2 \sin x + \log_2^2 \sin x + \log_2^3 \sin x + \dots = \frac{2}{3};$
- 4) $\log x + \frac{1}{2} \log x + \frac{1}{4} \log x + \dots = 2;$
- 5) $\log x + \log \sqrt[3]{x} + \log \sqrt[9]{x} + \dots = 3;$
- 6) $1 - \sin x + \sin^2 x - \sin^3 x + \dots = \frac{2}{3};$
- 7) $8^{1+|\cos x|+\cos^2 x+\dots} = 64.$

Rješenje. 1)

$$\begin{aligned} 2^{x+x^2+x^3+\dots} &= 2\sqrt{2}, \\ 2^{x+x^2+x^3+\dots} &= 2^{\frac{3}{2}}, \\ x + x^2 + x^3 + \dots &= \frac{3}{2}; \end{aligned}$$

Na lijevoj strani je geometrijski red koji je očito konvergentan ($|q| = |x| < 1$), pa imamo:

$$\begin{aligned} \frac{x}{1-x} &= \frac{3}{2}, \\ 2x &= 3 - 3x, \\ x &= \frac{3}{5}; \end{aligned}$$

2) $x^{\log(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots)} = 1;$

Sredimo eksponent na lijevoj strani:

$$\log\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots\right) = \log \frac{\frac{1}{2}}{1 - \frac{1}{2}} = \log 1 = 0.$$

Sada dani izraz prelazi u:

$$x^0 = 1,$$

te je rješenje polazne jednadžbe $\forall x \in \mathbf{R} \setminus \{0\};$

$$3) 1 + \log_2 \sin x + \log_2^2 \sin x + \log_2^3 \sin x + \dots = \frac{2}{3};$$

Uvjeti:

$$\sin x > 0 \implies x \in \langle 2k\pi, \pi + 2k\pi \rangle, k \in \mathbf{Z},$$

Na lijevoj strani je konvergentan geometrijski red ako je $|q| = |\log_2 \sin x| < 1$, tj.

$$\log_2 \sin x < 1 \implies \sin x < 2 \quad i$$

$$\log_2 \sin x > -1 \implies \sin x > \frac{1}{2} \implies x \in \left\langle \frac{\pi}{6} + 2k\pi, \frac{5\pi}{6} + 2k\pi \right\rangle;$$

Sada imamo:

$$\frac{1}{1 - \log_2 \sin x} = \frac{2}{3}$$

$$3 = 2 - 2 \log_2 \sin x$$

$$\log_2 \sin x = -\frac{1}{2}$$

$$\sin x = 2^{-\frac{1}{2}}$$

$$\sin x = \frac{1}{\sqrt{2}}$$

$$\sin x = \frac{\sqrt{2}}{2};$$

$$x_1 = \frac{\pi}{4} + 2k\pi, \quad x_2 = \frac{3\pi}{4} + 2k\pi, \quad k \in \mathbf{Z};$$

$$4) \log x + \frac{1}{2} \log x + \frac{1}{4} \log x + \dots = 2;$$

Uvjet: $x > 0$.

Na lijevoj strani je konvergentan geometrijski red $\left(q = \frac{1}{2}\right)$ pa imamo:

$$\log x + \frac{1}{2} \log x + \frac{1}{4} \log x + \dots = 2$$

$$\frac{\log x}{1 - \frac{1}{2}} = 2$$

$$2 \log x = 2$$

$$\log x = 1$$

$$x = 10;$$

$$5) \log x + \log \sqrt[3]{x} + \log \sqrt[9]{x} + \dots = \log x + \frac{1}{3} \log x + \frac{1}{9} \log x + \dots;$$

Uvjet: $x > 0$.

Na lijevoj strani je konvergentan geometrijski red $\left(q = \frac{1}{3}\right)$ pa imamo:

$$\begin{aligned} \log x + \frac{1}{3} \log x + \frac{1}{9} \log x + \dots &= 3 \\ \frac{\log x}{1 - \frac{1}{3}} &= 3 \\ \frac{3 \log x}{2} &= 3 \\ \log x &= 2 \\ x &= 10^2 \\ x &= 100; \end{aligned}$$

6) $1 - \sin x + \sin^2 x - \sin^3 x + \dots = \frac{2}{3}$

$$q = -\sin x, |q| < 1 \implies \sin x \neq \pm 1 \implies \sin x \notin \left\{ \frac{\pi}{2} + 2k\pi, \frac{3\pi}{2} + 2k\pi, k \in \mathbf{Z} \right\};$$

Sada imamo:

$$\begin{aligned} 1 - \sin x + \sin^2 x - \sin^3 x + \dots &= \frac{2}{3} \\ \frac{1}{1 + \sin x} &= \frac{2}{3} \\ 3 &= 2 + 2 \sin x \\ \sin x &= \frac{1}{2} \\ x &= (-1)^n \frac{\pi}{6} + n\pi, \quad n \in \mathbf{Z}; \end{aligned}$$

7)

$$8^{1+|\cos x|+\cos^2 x+\dots} = 64$$

$$8^{1+|\cos x|+\cos^2 x+\dots} = 8^2$$

$$1 + |\cos x| + \cos^2 x + \dots = 2$$

$$\frac{1}{1 - |\cos x|} = 2$$

$$1 - |\cos x| = \frac{1}{2}$$

$$|\cos x| = \frac{1}{2}$$

$$\cos x = \pm \frac{1}{2}$$

$$x = \pm \frac{\pi}{3} + k\pi, \quad k \in \mathbf{Z}.$$