

Zadatak 46. Izračunaj limes nizova s općim članom:

$$1) a_n = \left(\frac{1}{n^2} + \frac{2}{n^2} + \dots + \frac{n-1}{n^2} + \frac{n}{n^2} \right);$$

$$2) a_n = \left(\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{n(n+1)} \right);$$

$$3) a_n = \frac{1^2 + 2^2 + \dots + n^2}{n^3}.$$

Rješenje. 1) Vrijedi $1 + 2 + \dots + n = \frac{n(n+1)}{2}$, zato je

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n^2 + n}{2n^2} = \lim_{n \rightarrow \infty} \left(\frac{1}{2} + \frac{1}{2n} \right) = \frac{1}{2}.$$

2) Sada je

$$\begin{aligned} \lim_{n \rightarrow \infty} a_n &= \lim_{n \rightarrow \infty} \left(\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{n(n+1)} \right) \\ &= \lim_{n \rightarrow \infty} \left(\left(1 - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \dots + \left(\frac{1}{n-1} - \frac{1}{n} \right) \right) \\ &= \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n} \right) = 1. \end{aligned}$$

3) Zbroj brojeva u brojniku je $\frac{1}{6}n(n+1)(2n+1)$. Zato je limes niza:

$$\begin{aligned} \lim_{n \rightarrow \infty} a_n &= \lim_{n \rightarrow \infty} \frac{\frac{1}{6}n(n+1)(2n+1)}{n^3} = \lim_{n \rightarrow \infty} \frac{2n^3 + 3n^2 + n}{6n^3} \\ &= \lim_{n \rightarrow \infty} \left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2} \right) = \frac{1}{3}. \end{aligned}$$