Zadatak 48. Nađi limes sljedećih nizova:

1) $a_1 = 3\,,\ a_{n+1} = \dfrac{a_n + 1}{2}\,,\ n > 1\,;$

2) $a_1 = 1\,,\ a_2 = \frac{5}{2}\,,\ a_n = \frac{3}{2}a_{n-1} - \frac{1}{2}a_{n-2}\,,$
$n > 2\,;$

3) $a_1 = \sqrt{3}\,,\ a_{n+1} = \sqrt{3 + a_n}\,,\ n > 1\,;$

4) $a_1 = 2\,,\ a_{n+1} = \frac{1}{3}a_n + \frac{1}{3}\,,\ n > 1\,.$

Rješenje. **1)**

$$\left.\begin{aligned} a_1 &= 3 = 1 + 2 \\ a_2 &= 2 = 1 + 1 \\ a_3 &= \frac{3}{2} = 1 + \frac{1}{2} \\ a_4 &= \frac{5}{4} = 1 + \frac{1}{4} \\ a_5 &= \frac{9}{8} = 1 + \frac{1}{8} \end{aligned}\right\} \implies \begin{aligned} &a_n = 1 + \frac{1}{2^{n-2}}, \\ &\lim_{n\to\infty} a_n = 1; \end{aligned}$$

2)

$$\left.\begin{aligned} a_1 &= 4 - 3 \\ a_2 &= \frac{5}{2} = 2 + \frac{1}{2} = 4 - \frac{3}{2} \\ a_3 &= \frac{3}{2}\cdot\frac{5}{2} - \frac{1}{2} = \frac{13}{4} = 4 - \frac{3}{4} \\ a_4 &= \frac{3}{2}\cdot\frac{13}{4} - \frac{1}{2}\cdot\frac{5}{2} = \frac{29}{8} = 4 - \frac{3}{8} \\ a_5 &= \frac{3}{2}\cdot\frac{29}{8} - \frac{1}{2}\cdot\frac{13}{4} = \frac{61}{16} = 4 - \frac{3}{16} \end{aligned}\right\} \implies \begin{aligned} &a_n = 4 - \frac{1}{2^{n-1}}, \\ &\lim_{n\to\infty} a_n = 4; \end{aligned}$$

3)

$$\begin{aligned} a_1 &= \sqrt{3} \\ a_2 &= \sqrt{3 + \sqrt{3}} \\ a_3 &= \sqrt{3 + \sqrt{3 + \sqrt{3}}} \\ a_4 &= \sqrt{3 + \sqrt{3 + \sqrt{3 + \sqrt{3}}}}; \end{aligned}$$

Niz je rastući i

$$a_{n+1} = \sqrt{3 + a_n} < \sqrt{3 + 3} = \sqrt{6}$$

Dakle niz je rastući i omeđen pa ima limes a. Sad u jednakosti

$$a_{n+1} = \sqrt{3 + a_n}$$

možemo pustiti da n teži u beskonačnost:

$$a = \sqrt{3 + a},$$

odakle slijedi:

$$a^2 - a - 3 = 0, \quad a_{1,2} = \frac{1 \pm \sqrt{13}}{2};$$

Rješenje ove jednadžbe koje zadovoljava uvijet $a = \sqrt{3+a}$ je

$$\frac{\sqrt{13}+1}{2};$$

4)

$$\begin{aligned}
a_1 &= 2\\
a_2 &= \frac{1}{3}(2+1) = 1\\
a_3 &= \frac{1}{3}(1+1) = \frac{2}{3}\\
a_4 &= \frac{1}{3}\left(\frac{2}{3}+1\right) = \frac{5}{9}\\
a_5 &= \frac{1}{3}\left(\frac{5}{9}+1\right) = \frac{14}{27}
\end{aligned}$$

Može se naslutiti da je a_n oblika $x \cdot \left(\frac{1}{3}\right)^{n-2} + y$, $x, y \in \mathbf{R}$. Nađimo x, y uvrstivši vrijednosti za a_1, a_2:

$$\left.\begin{aligned} 2 &= x \cdot 3 + y\\ 1 &= x \cdot 1 + y \end{aligned}\right\} - \implies 1 = 2x, \ \underline{x = \frac{1}{2}}, \ \underline{y = \frac{1}{2}};$$

pa je

$$a_n = \frac{1}{2} \cdot \left(\frac{1}{3}\right)^{n-2} + \frac{1}{2}$$

i

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{1}{2} \cdot \left(\frac{1}{3}\right)^{n-2} + \frac{1}{2}\right) = \frac{1}{2}.$$